Limits at infinity - Consider \( \lim_{x \to \infty} f(x) = L \). This means that \( f(x) \) can be made arbitrarily close to \( L \) by taking a large enough \( x \).

\[
\lim_{x \to \infty} \frac{1}{x} = 0
\]

For any function \( f(x) = \frac{1}{x^n} \) where \( n \) is a natural number:

\[
\lim_{x \to \pm \infty} f(x) = 0 \quad \text{provided} \quad f(x) \text{ is defined}
\]

Example:

\[
\lim_{x \to \infty} \frac{x^2 + 3}{2x^3 + 1}
\]

Bad notation: \( \frac{(\infty)^2 + 3}{2(\infty)^3 + 1} \) → \( \infty \) → \( \infty \)

Then

\[
\lim_{x \to \infty} \frac{1}{x^3} \cdot \frac{x^2 + 3}{2x^3 + 1} = \lim_{x \to \infty} \frac{\frac{1}{x^3} + \frac{3}{x^3}}{2 + \frac{1}{x^3}}
\]

Used limit rules:

\[
\lim_{x \to \infty} \frac{1}{x} + \lim_{x \to \infty} \frac{3}{x^3} = \frac{0 + 0}{2 + 0} = 0
\]

\[
\lim_{x \to \infty} 2 + \lim_{x \to \infty} \frac{1}{x^3} = \lim_{x \to \infty} \frac{1}{x} + \lim_{x \to \infty} \frac{3}{x^3} = \frac{0 + 0}{2 + 0} = 0
\]
\[ f(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a}{b_m x^m + b_{m-1} x^{m-1} + \ldots + b_1 x + b} \]

If \( m > n \): \( \lim_{x \to \pm \infty} f(x) = 0 \)

If \( n > m \): \( \lim_{x \to \pm \infty} f(x) \) Does not exist

If \( n = m \): \( \lim_{x \to \pm \infty} f(x) = \frac{a_n}{b_m} \)

\[ \frac{3x^4 + 4x^2 - x + 2}{5x^2 + 3x^2 - 4} \]

\( = \text{D.N.E.} \)

\[ \frac{5x^3 - 4x + 2}{7x^3 + 5x^2 - 6} = \frac{5}{7} \]

**Ex:**

\( f(x) = \begin{cases} 
-x + 2, & x < 0 \\
\sqrt{x}, & x \geq 0 
\end{cases} \)

\( \lim_{x \to 0^+} f(x) = 0 \)

\( \lim_{x \to 0^-} f(x) = 2 \)

**One-sided limits**

Sometimes we will need to take a one-sided limit.

**Method 2**

Our limits are approaching different values.
Problem 1

\[ f(x) = \begin{cases} 
  x, & x \geq 1 \\
  x^2 - 1, & x < 1 
\end{cases} \]

\[ \lim_{x \to 1^+} f(x) = 1 \]
\[ \lim_{x \to 1^-} f(x) = 0 \]

Problem 2

\[ g(x) = \sqrt{x} \]

\[ \lim_{x \to 4^-} g(x) = 2 \]
\[ \lim_{x \to 4^+} g(x) = 2 \]

Let \( f \) be a function defined on some interval containing a real number \( c \), except possibly at \( c \).

Then \( \lim_{x \to c} f(x) = L \) if and only if \( \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L \).

If the two-sided limits equal each other then it exists.
If the two-sided limits don't equal each other, then overall limit \( \text{DNE} \).

By Problem 2 from above

\[ \lim_{x \to 4} g(x) = 2 \]

Continuous functions - (you can take derivatives if function is continuous)

A function \( f \) is continuous at a point \((c, f(c))\) if

1. \( f(c) \) is defined
2. \( \lim_{x \to c} f(x) \) exist (both sides)
3. \( \lim_{x \to c} f(x) = f(c) \)
ex) \( f(x) = \begin{cases} 2, & x = 0 \\ x^2, & x \neq 0 \end{cases} \)

Is it continuous at 0?

1. \( f(0) = 2 \) **defined**
2. \( \lim_{x \to 0} f(x) = 0 \) **exist**
3. 0 \( \neq 2 \) not continuous

ex2) \( g(x) = \begin{cases} 2, & x = 0 \\ \frac{1}{x}, & x \neq 0 \end{cases} \)

Is it continuous at \( x = 2 \)?

1. \( f(2) = \frac{1}{2} \) **defined**
2. \( \lim_{x \to 2} g(x) = \frac{1}{2} \) **exist**
3. \( \frac{1}{2} = \frac{1}{2} \) continuous

\[ \lim_{x \to 2} g(x) = g(2) \] It is continuous at 2.

A function is continuous at an interval if it is continuous at each point in that interval.

A function is continuous on its domain if it is continuous at each point in its domain.

\[ f(x) = (x+2)^2 + 2 \] where is this function continuous everywhere.

Properties of continuous functions

1. \( f(x) = c \) is continuous everywhere
2. \( f(x) = x \) is continuous everywhere
3. If \( f \) and \( g \) are both continuous at \( x = c \), then \( [f(x)]^n \) is continuous at \( x = c \)
4. \( f \pm g \) is continuous at \( x = c \)
5. \( f \cdot g \) is continuous at \( x = c \)
6. \( \frac{f}{g} \) is continuous at \( x = c \)
7. Polynomials are continuous everywhere.
8. Rational functions \( \frac{p(x)}{q(x)} \) are continuous everywhere provided \( q(x) \neq 0 \)