04-08-2019

• **Ex**: Finding Taylor series for $f(x)=\ln(3+9x)$ around $x=2$.

→ We found: $n=0 \quad f(2)=\ln(21)$

$n=1,2,\dots \quad f^{(n)}(2)=(-1)^{n+1}(n-1)!\left(\frac{3}{7}\right)^n$

Then:

$$\ln(3+9x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(2)}{n!}(x-a)^n=\underbrace{\frac{f(2)}{0!}(x-2)^0}_{n=0}+\sum_{n=1}^{\infty}\frac{f^{(n)}(2)}{n!}(x-a)^n$$

$$=f(2)+\sum_{n=1}^{\infty}\frac{f^{(n)}(2)}{n!}(x-a)^n$$

$$=\ln(21)+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cancel{(n-1)!}\left(\frac{3}{7}\right)^n(x-2)^n}{\underset{n\cancel{(n-1)!}}{n!}}$$

$$\ln(3+9x)=\ln(21)+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(\frac{3}{7}\right)^n(x-2)^n$$

Ex: Taylor Series for $f(x)=\frac{7}{x^3}$ about $x=-4$.

$n=0 \quad f(x)=7x^{-3}$

$n=1 \quad f'(x)=7(-3)x^{-4}$

$n=2 \quad f''(x)=7(3)(4)x^{-5}$

$n=3 \quad f'''(x)=7(3)(4)(-5)x^{-6}$

$n=4 \quad f^{IV}(x)=7(3)(4)(5)(6)x^{-7}$

$\frac{2}{2}$

Thus: $f^{(n)}(x)=(-1)^n(7)\frac{(n+2)!}{2}x^{-(n+3)}$

or

$=(-1)^n\left(\frac{7}{2}\right)(n+2)!\,x^{-(n+3)}$

Notice this formula works for all $n=0,1,2,3,\dots$

then:

$$f^{(n)}(-4)=(-1)^n\left(\frac{7}{2}\right)(n+2)!\,(-4)^{-(n+3)}$$

and:

$$f(x)=\sum_{n=0}^{\infty}(-1)^n\left(\frac{7}{2}\right)\overset{(n+2)(n+1)\cancel{n!}}{(n+2)!}(-4)^{-(n+3)}\frac{1}{\cancel{n!}}(x+4)^n$$

$$=\sum_{n=0}^{\infty}\cancel{(-1)^n}\left(\frac{7}{2}\right)\frac{(n+2)(n+1)}{\underset{(-1)^{n+3}(4)^{n+3}}{(-4)^{n+3}}}(x+4)^n$$

$$=\sum_{n=0}^{\infty}\left(-\frac{7}{2}\right)\frac{(n+2)(n+1)}{4^{n+3}}(x+4)^n$$

$4^3\cdot 4^n$

$2^6\cdot 4^n$

$$f(x)=\frac{7}{x^3}=\sum_{n=0}^{\infty}\frac{-7}{2^7}(n+2)(n+1)\left(\frac{x+4}{4}\right)^n$$

• **Application:** integrating the impossible.

$$\int \frac{\sin(x)}{x}\,dx = \int \underbrace{\frac{1}{x}\sin(x)}_{\text{series at } x=0}\,dx = \int \frac{1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}\,dx$$

$$= \int \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!}\,dx$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!\,(2n+1)} + C$$