04-04-2019

Taylor Series: provides us with a way to write a function as a power series.

power series: $f(x)=\sum_{n=0}^{\infty} C_n(x-a)^n = C_0 + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^3 + C_4(x-a)^4 + \dots$

NOTICE: $f(a)=C_0$, what if I want C_1?

$f'(x)= C_1 + 2C_2(x-a) + 3C_3(x-a)^2 + 4C_4(x-a)^3 + \dots$

then $f'(a)=C_1$

$f''(x)= 2C_2 + 3\cdot 2C_3(x-a) + 4\cdot 3C_4(x-a)^2 + 5\cdot 4C_5(x-a)^3 + \dots$

then $f''(a)=2C_2$

$f'''(x)= 3\cdot 2\cdot C_3 + 4\cdot 3\cdot 2C_4(x-a) + 5\cdot 4\cdot 3C_5(x-a)^2 + \dots$

then $f'''(a)= 3\cdot 2\cdot C_3$

$f^{IV}(x)= 4\cdot 3\cdot 2C_4 + 5\cdot 4\cdot 3\cdot 2(x-a) + 6\cdot 5\cdot 4\cdot 3(x-a)^2 + \dots$

$f^{IV}(a)= 4\cdot 3\cdot 2\cdot C_4$

Then, how are C_i's being obtained?

$C_0=f(a)$, $C_1=f'(a)$, $C_2=\frac{f''(a)}{2}$, $C_3=\frac{f'''(a)}{3\cdot 2}$, $C_4=\frac{f^{(4)}(a)}{4\cdot 3\cdot 2}$, ...

Looks like the pattern is:

$$C_n=\frac{f^{(n)}(a)}{n!}$$

▶ The Taylor Series for $f(x)$ about $x=a$ is: Thus,

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

▶ The McClaurin series for $f(x)$ is: centered at zero ($a=0$).

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$$

Ex: Find Taylor Series for $e^x=f(x)$.

→ Notice $f^{(n)}(x)=e^x$ for $n=0,1,2,\dots$ and $a=0$ = zero.

$f^{(n)}(0)=e^0=1$

Then: $e^x=\sum_{n=0}^{\infty}\frac{1}{n!}x^n$. - - - - - - - - - - - - - - - - - →

Remember: about $x=0$. $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$

Ex: Taylor Series for $f(x)=e^{-x}$ about $x=0$.

$n=0$ $f(x)=e^x$

$n=1$ $f'(x)=-e^x$

$n=2$ $f''(x)=e^x$

$n=3$ $f'''(x)=-e^x$

pattern:

$f^{(n)}(x)=(-1)^n e^{-x}$ → $f^{(n)}(0)=(-1)^n$ for $x=0$

for $n=0,1,2,3,\dots$

Thus the series is given by:

$$f(x) = e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n$$

$$e^p = \sum_{n=0}^{\infty} \frac{p^n}{n!}$$

A way to quickly solve this type of problem could be:
Thus, for $f(x) = e^{-x}$

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n$$

<u>Ex</u>: $f(x) = 3x^2 e^{-4x^{10}}$. Using our trick:

$$3x^2 e^{-4x^{10}} = 3x^2 \sum_{n=0}^{\infty} \frac{(-4x^{10})^n}{n!}$$

$$= 3x^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (4)^n x^{10n}$$

$$= \sum_{n=0}^{\infty} 3 \frac{(-1)^n}{n!} (4)^n x^{10n+2}$$

Remember:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \qquad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Ex: Write the Taylor Series for $f(x) = e^{2x}$ about -1.

$n=0$	$f(x) = e^{2x}$	e^{-2}
$n=1$	$f'(x) = 2e^{2x}$	$2e^{-2}$
$n=2$	$f''(x) = 4e^{2x}$	$4e^{-2}$
$\vdots$	$\vdots$	$\vdots$
n	$f^{(n)}(x) = 2^n e^{2x}$	$2^n e^{-2}$
		$f^{(n)}(-1)$

Then:

$$f(x) = \sum_{n=0}^{\infty} \frac{2^n e^{-2}}{n!}(x+1)^n$$

Ex: Taylor Series for $f(x) = \cos(x)$ about $x=0$.

$n=0$	$f(x) = \cos x$	1
$n=1$	$f'(x) = -\sin x$	0
$n=2$	$f''(x) = -\cos x$	-1
$n=3$	$f^{(3)}(x) = \sin x$	0
$n=4$	$f^{(4)}(x) = \cos$	1
$n=5$	$f^{(5)}(x) = -\sin x$	0
$n=6$	$f^{(6)}(x) = -\cos x$	-1
$n=7$	$f^{(7)}(x) = \sin x$	0

$$\cos(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$= 1 + 0 - \frac{x^2}{2!} + 0 + \frac{x^4}{4!} + 0 - \frac{x^6}{6!} + \dots$$

$$= \underset{k=0}{1} - \underset{k=1}{\frac{x^2}{2!}} + \underset{k=2}{\frac{x^4}{4!}} - \underset{k=3}{\frac{x^6}{6!}} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} - \frac{x^{14}}{14!}$$

$\dots \quad k=k$?

$$(-1)^k \frac{x^{2k}}{(2k)!}$$

Thus:

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Ex: What about $f(x) = \sin(x)$?

$n=0$	$\sin x$	0
$n=1$	$\cos x$	1
$n=2$	$-\sin x$	0
$n=3$	$-\cos x$	-1
$n=4$	$\sin x$	0

$$\sin(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$= 0 + \frac{x}{1!} + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + 0 - \frac{x^7}{7!} + \dots$$

$$= \underset{k=0}{\frac{x}{1!}} - \underset{k=1}{\frac{x^3}{3!}} + \underset{k=2}{\frac{x^5}{5!}} - \underset{k=3}{\frac{x^7}{7!}} + \underset{k=4}{\frac{x^9}{9!}} - \underset{k=5}{\frac{x^{11}}{11!}} + \dots \quad k=k$$

$$\frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

Thus,

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

<u>Ex:</u> Taylor series for $f(x) = \ln(3+9x)$ about $x=2$.

$n=0 \quad \ln(3+9x)$

$n=1 \quad \frac{1}{3+9x}(9) = 9(3+9x)^{-1}$

$n=2 \quad -9^2(3+9x)^{-2}$

$n=3 \quad 9^3(2)(3+9x)^{-3}$

$n=4 \quad -9^4(2)(3)(3+9x)^{-4}$

$n=5 \quad 9^5(4)(3)(2)(3+9x)^{-5}$

$f^{(n)}(x) = (-1)^{n+1} 9^n (n-1)!\,(3+9x)^{-n}$ not at $n=0$ $n=1,2,3,\ldots$

for $n=1,2,3\ldots$ $f^{(n)}(2) = (-1)^{n+1} 9^n (n-1)!\,(21)^{-n}$

$= \frac{(-1)^{n+1}\, 9^n\,(n-1)!}{21^n}$

$n=0 \longrightarrow f^{(0)}(2) = \ln(3+9(2)) = \ln 21$