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Power Series and Functions:

⇒ Remember Geometric series:

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \quad \text{provided } |r|<1 \qquad \text{let: } a=1,\ r=x \qquad \text{Then } \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \text{ provided } |x|<1$$

Ex: Find the series representation for:

$$f(x) = \frac{1}{1+x^3} = \frac{1}{1-(-x^3)} = \sum_{n=0}^{\infty} (-x^3)^n \quad \text{provided } |-x^3|<1$$

$$|x|^3<1$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{3n} \quad \text{provided } |x|<1$$

Notice: centered at zero, radius of conv. of 1, $-1<x<1$ (guaranteed convergence).

Ex: $f(x) = \frac{4x^2}{1+x^3} = 4x^2 \cdot \frac{1}{1+x^3}$

$$= 4x^2 \sum_{n=0}^{\infty} (-1)^n x^{3n} \quad \text{provided } |x|<1$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{3n} 4x^2 = \sum_{n=0}^{\infty} (-1)^n 4x^{3n+2} \quad \text{given that } |x|<1.$$

Ex: $g(x) = \frac{x}{5-x} = \frac{x}{5} \cdot \frac{1}{1-\frac{x}{5}} = \frac{x}{5} \sum_{n=0}^{\infty} \left(\frac{x}{5}\right)^n$ provided $\left|\frac{x}{5}\right|<1 \cdots \frac{|x|}{|5|}<1$

$|x|<5$

$$= \sum_{n=0}^{\infty} \left(\frac{x}{5}\right)^{n+1} \text{ or } \sum_{n=0}^{\infty} \frac{x^{n+1}}{5^{n+1}} \quad \text{provided that } |x|<5$$

Notice here we have a radius of guaranteed convergence of 5.

Unfortunately, there are not that many fxns that can be represented as series using these tricks and manipulations.

Observe:

if: $f(x) = \sum_{n=0}^{\infty} C_n (x-a)^n = C_0 + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^3 + \ldots$

then: $f'(x) = \ldots = 0 + C_1 + 2C_2(x-a) + 3C_3(x-a)^2 + \ldots$

$$\sum_{n=0}^{\infty} C_n n \cdot (x-a)^{n-1}$$

need to be able to see patterns...

or $n=1 \longrightarrow n=0$

But actually we can just take the der (wrt x) in the n^{th} term of the series.

Also:

$$f''(x) = \sum_{n=0}^{\infty} n(n-1) C_n (x-a)^{n-2}$$

Powerseries representation for an INTEGRAL:

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$$

How is this useful?

Ex: • $f(x) = \frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{-1}{1-x}\right) = \frac{d}{dx}\left[-\sum_{n=0}^{\infty} x^n\right]$

$$= -\sum_{n=0}^{\infty} n x^{n-1}$$

• $f(x) = \ln(5-x) = -\int \frac{1}{5-x}\,dx = -\frac{1}{5}\int \frac{1}{1-\frac{x}{5}}\,dx$

$$= -\frac{1}{5}\int \sum_{n=0}^{\infty} \left(\frac{x}{5}\right)^n dx$$

$$= -\frac{1}{5}\int \sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n x^n\,dx$$

$$= \left[-\frac{1}{5}\sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n \frac{1}{n+1} x^{n+1}\right] + C$$