03-07-2019

Series (cont'd):

- Stripping terms out of a series:

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + \dots$$
$$= a_0 + \sum_{n=1}^{\infty} a_n$$

Remember series are not exactly big grant sums...

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + \sum_{n=2}^{\infty} a_n$$

↓ then conv. ← if conv ↓

if convergent ⟶ then convergent

Convergence/Divergence of a series:

▶ $$\sum_{i=1}^{\infty} a_i = \lim_{N\to\infty} \underbrace{\sum_{i=1}^{N} a_i}_{S_N}$$

S_N: partial sum $[S_N = a_1 + a_2 + \dots + a_N]$

Ex: Conv. or Diver.? If conv., find its value?

- $$\sum_{n=1}^{\infty} n$$

Partial sums:

$$S_N = \sum_{i=1}^{N} i \quad \text{, so } \lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{i=1}^{N} i = \lim_{N\to\infty} \frac{N(N+1)}{2} = \infty$$

So, divergent.

Ex:

- $$\sum_{n=2}^{\infty} \frac{1}{n^2-1} \to S_N = \sum_{i=2}^{N} \frac{1}{i^2-1} = \frac{3}{4} - \frac{1}{2N} - \frac{1}{2(N+1)}$$

$$\sum_{i=2}^{\infty} \frac{1}{i^2-1} = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(\frac{3}{4} - \frac{1}{2N} - \frac{1}{2(N+1)}\right) = \frac{3}{4}$$

So, convergent, and to $\frac{3}{4}$.

Ex:

- $$\sum_{n=0}^{\infty} (-1)^n \to S_N = \sum_{i=0}^{N} (-1)^i$$

$S_0 = 1$

$S_1 = 1-1=0$

$S_2 = 1-1+1 = 1$

$S_3 = 1-1+1-1 = 0$

$\vdots$

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{i=0}^{N} (-1)^i = DNE$$

The limit of this does not exist.

$S_N = \{1, 0, 1, 0, 1, 0, \dots\}$

So, divergent.

Ex: • $\sum_{n=1}^{\infty} \frac{1}{3^{n-1}} \rightarrow S_N = \sum_{i=1}^{N} \frac{1}{3^{i-1}} = \frac{3}{2}\left(1-\frac{1}{3^N}\right)$

$$\sum_{n=1}^{\infty} \frac{1}{3^{n-1}} = \lim_{N\to\infty} \sum_{i=1}^{N} \frac{1}{3^{i-1}} = \lim_{N\to\infty} \frac{3}{2}\left(1-\frac{1}{3^N}\right) = \frac{3}{2}$$

So, convergent and to 3/2.

Let's remember the problems we just saw and take their limits:

① diverged
$\lim_{n\to\infty} n = \infty$

② converged
$\lim_{n\to\infty} \frac{1}{n^2-1} = 0$

③ diverged
$\lim_{n\to\infty} (-1)^n = \text{DNE}$

④ Converged
$\lim_{n\to\infty} \frac{1}{3^{n-1}} = 0$

Illustrates
if $\sum a_n = L$ for some L, then $\lim_{n\to\infty} a_n = 0$.

FACT:
If a series $\sum a_i$ converges, then $\lim_{n\to\infty} a_n = 0$

Don't ever think it's true the other way around!!!

FACT (contrap.): If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_i$ diverges.

Divergence Tests:

• Given $\sum a_n$, if $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.

Ex: $\sum_{n=4}^{\infty} \frac{3n^6-4n+2}{7n^6+3n^4+10}$

Note: $\lim_{n\to\infty} \frac{3n^6-4n+2}{7n^6+3n^4+10} = \frac{3}{7} \neq 0$. Then this series diverges.

→ Special: **GEOMETRIC SERIES:**

Any series of the form $\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n$

It'll converge if $|r| < 1$ -------> $\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$

It'll diverge if $|r| \geq 1$

Ex:

• $\sum_{n=1}^{\infty} 9^{-n+2} 4^{n+1}$ Start by looking at the starting index.

$$= \sum_{n=1}^{\infty} \frac{4^{n+1}}{9^{n-2}} = \sum_{n=1}^{\infty} \frac{4^{n+1-1+1}}{9^{n-1-1}} = \sum_{n=1}^{\infty} \frac{4^{n-1}\cdot 4^2}{9^{n-1}\cdot 9^{-1}} = \sum_{n=1}^{\infty} \underbrace{(4^2\cdot 9)}_{a} \underbrace{\left(\frac{4}{9}\right)}_{r}^{n-1}$$

$|r| = \left|\frac{4}{9}\right| = \frac{4}{9} < 1$