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Ex: $\sum_{n=2}^{\infty} \frac{n^2}{(2n-1)!}$ Ratio test:

$$L = \lim_{n\to\infty} \left| \frac{(n+1)^2}{(2n+1)!} \cdot \frac{(2n-1)!}{n^2} \right| = \lim_{n\to\infty} \frac{(n+1)^2 \cancel{(2n-1)!}}{(2n+1)(2n)\cancel{(2n-1)!}\, n^2}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{(2n+1)\, 2 \cdot n^3} = 0 \text{ (higher degree in the denominator).}$$

$0 < 1$, then the series converges.

Ex: $\sum_{n=0}^{\infty} \frac{(-1)^{n+3}}{4n+1}$

$$L = \lim_{n\to\infty} \left| \frac{\cancel{(-1)^{n+4}}}{4n+5} \cdot \frac{4n+1}{\cancel{(-1)^{n+3}}} \right|$$

$$= \lim_{n\to\infty} \frac{4n+1}{4n+5} = 1 = 1$$

Then the ratio test tells us nothing.

- Alternating Series: $a_n = \frac{1}{4n+1} > 0$

☑ decreasing

☑ $\lim_{n\to\infty} \frac{1}{4n+1} = 0$

Then the series converges (by Altern. Series).

Root Test:

Given $\sum a_n$, compute:

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{1/n}$$

1. If $L < 1$, the series is absolutely convergent.
2. If $L > 1$, the series is divergent.
3. If $L = 1$, the test is inconclusive.

Ex: $\sum_{n=1}^{\infty} \frac{n^n}{4^{3n+1}}$ → Root test:

$$L = \lim_{n\to\infty} \left| \frac{n^n}{4^{3n+1}} \right|^{1/n} = \lim_{n\to\infty} \frac{(n^n)^{1/n}}{(4^{3n+1})^{1/n}} = \lim_{n\to\infty} \frac{n}{4^{3+\frac{1}{n}}} = \frac{\infty}{4^3} = \infty$$

Clearly $\infty > 1$

Then the series diverges by the root test.

Ex: $\sum_{n=0}^{\infty}\left(\frac{1-6n}{4+10n}\right)^n$ Root Test:

$$L=\lim_{n\to\infty}\left|\left(\frac{1-6n}{4+10n}\right)^n\right|^{\frac{1}{n}}=\lim_{n\to\infty}\left|\frac{1-6n}{4+10n}\right|$$

$$=\left|\frac{-6}{10}\right|=\frac{3}{5}<1$$

Then we found that this series converges.

Strategy for series (summary):

1. $\lim_{n\to\infty} a_n \neq 0$ Divergence Test
2. Geometric series or P-series ($\sum \frac{a}{n^p}$)
3. $\frac{\text{Poly}}{\text{Poly}}$ Comparisor or Limit Comparison Test ← also look for $\sin^2, \cos^2$
4. Alternating Series? $(-1)^n$, $(-1)^{n+1}$
5. Factorial : Ratio Test
   Geometric (messy)
   Vaguely Geometric ex: $\frac{(-4)^{n+1}}{3^n(n+1)}$
6. $(\ldots)^n$ Geometric (sometimes)
   Root Test
7. Integral Test: don't do it until you really have to.
8. If I ask for a series value?
   → Divergent → Geometric → Telescoping