03-22-2019

- Testing the Limit Comparison Test:

Consider the following series...

$\sum_{n=1}^{\infty} \frac{1}{n}$ (D)

or $\lim_{n\to\infty} \frac{1}{n^2} \cdot \frac{n}{1} = \frac{1}{n} = 0$ They don't share

$\lim_{n\to\infty} \frac{1}{n} \cdot \frac{n^2}{1} = \lim_{n\to\infty} n = \infty$ the same convergence.

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ (C)

$\lim_{n\to\infty} \frac{1}{n^2} \cdot \frac{n^3}{1} = \lim_{n\to\infty} n = \infty$ But they both converge!

$\sum_{n=1}^{\infty} \frac{1}{n^3}$ (C)

∞ or 0 don't tell us anything, just that we made the wrong choice.

Alternating Series Test:

$\sum_{n=1}^{\infty} b_n$ where $b_n = (-1)^n a_n$ / $b_n = (-1)^{n+1} a_n$ where $a_n > 0$ for all n (or eventually).

Don't worry if you see...

$(-1)^{n+6} = (-1)^n (-1)^6 = (-1)^n$ or

$(-1)^{n-3} = (-1)^{n+1}(-1)^{-4} = (-1)^{n+1}$

- **TEST**:

Given $\sum b_n$ where $b_n = (-1)^n a_n$ or $b_n = (-1)^{n+1} a_n$ with $a_n > 0$ for all n.

Then if: ① $\lim_{n\to\infty} a_n = 0$ and ② a_n are decreasing for all n.

Then $\sum b_n$ is CONVERGENT.

NOTE: This test doesn't give DIVERGENT as a result.

Ex:

$\sum_{n=1}^{\infty} \frac{(-1)^{n+2}}{n}$ → Note that we have $a_n = \frac{1}{n} > 0$

① $\lim_{n\to\infty} \frac{1}{n} = 0$

② $a_n = \frac{1}{n}$ is always decreasing (clearly).

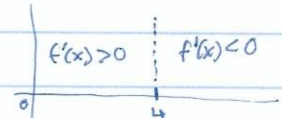
So, by alternating series test, this series converges.

Ex:

$\sum_{n=0}^{\infty} \frac{(-1)^{n+7}\sqrt{n}}{n+4}$ → Note that we have $a_n = \frac{\sqrt{n}}{n+4} > 0$

① $\lim_{n\to\infty} \frac{\sqrt{n}}{n+4} = 0$ since $deg(n+4) > deg(\sqrt{n})$

② Is $\frac{\sqrt{n}}{n+4}$ decreasing? $f(x) = \frac{\sqrt{x}}{x+4} \to f'(x) = \frac{4-x}{2\sqrt{x}(x+4)^2}$



Thus, the series eventually is decreasing.

Then, by the Alternating Series Test, this series converges.

Ex: $\sum_{n=4}^{\infty} \frac{(-1)^n n^2}{n^2+2}$ Note that $a_n = \frac{n^2}{n^2+2} > 0$.

① $\lim_{n\to\infty} \frac{n^2}{n^2+2} = 1 \neq 0$

Here, test says nothing, it just cannot be used.

How do we proceed now?

Try divergence test:

$$\lim_{n\to\infty} \frac{(-1)^n n^2}{n^2+2} = \lim_{n\to\infty} \left[(-1)^n \cdot \frac{n^2}{n^2+2} \right]$$

$$\neq \left(\underbrace{\lim_{n\to\infty} (-1)^n}_{-1 \leftrightarrow +1} \right) \left(\underbrace{\lim_{n\to\infty} \frac{n^2}{n^2+2}}_{\to 1} \right)$$

NOT ALWAYS TRUE, you can only split if you know the two parts converge. But let's see what's going on

$$= \text{DNE} \neq 0$$

Thus, by divergence test, the series diverges.

Absolute Convergence:

Given a series $\sum a_n$

① If $\sum |a_n|$ converges, then $\sum a_n$ is called absolutely convergent, and furthermore it's convergent.

② If $\sum |a_n|$ is divergent but we know that $\sum a_n$ converges, then $\sum a_n$ is said to CONDITIONALLY CONVERGE.

Ex:

We learnt that $\sum \frac{(-1)^{n+2}}{n}$ converges. But notice:

$$\sum \left| \frac{(-1)^{n+2}}{n} \right| = \sum \frac{|(-1)^{n+2}|}{|n|} = \sum \frac{1}{n} \text{ diverges.}$$

Then $\sum \frac{(-1)^{n+2}}{n}$ conditionally converges.