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Ex: • $\sum_{n=1}^{\infty} \frac{n}{n^2-\cos^2(n)}$ Guess it diverges. Then we need a smaller series that we know diverges.

Notice $n^2-\cos^2(n) < n^2$. Then

$$\frac{n}{n^2-\cos^2(n)} > \frac{n}{n^2} = \frac{1}{n}.$$

And we know $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic).

So, by comparison test, the series $\sum_{n=1}^{\infty} \frac{n}{n^2-\cos^2 n}$ also diverges.

Ex:

• $\sum_{n=1}^{\infty} \frac{e^{-n}}{n+\cos^2(n)}$ Let's guess it'll converge. Then we need a larger series that we know converges.

$$\underbrace{\frac{e^{-n}}{n+\cos^2(n)} < \frac{e^{-n}}{n}}_{\text{since } n+\cos^2 n > n} \underbrace{\leq \frac{e^{-n}}{1}}_{n \geq 1} = e^{-n}$$

How do we prove now that $\sum_{n=1}^{\infty} e^{-n}$ converges?

- Integral test:

  e^{-n} is always decreasing and positive.

  Then: $\int_1^{\infty} e^{-x}\,dx = -\lim_{b\to\infty}\left[e^{-x}\right]_1^b = -\lim_{b\to\infty}\left[\frac{1}{e^b} - \frac{1}{e^1}\right]$

  $= \frac{1}{e}$ CONVERGES!

  Then $\sum_{n=1}^{\infty} e^{-n}$ converges by the Integral Test!

- Geometric series:

  $\sum_{n=1}^{\infty} e^{-n} = \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n$ where we have $\left|\frac{1}{e}\right| < 1$

  Then $\sum_{n=1}^{\infty} e^{-n}$ converges.

Either way, we found a larger series that converges.

Then by the Comparison Test, the original series converges as well.

Ex:

• $\sum_{n=3}^{\infty} \frac{n^2+2}{n^4+6}$ Let's guess it'll converge. We need again a larger convergent series.

$$\frac{n^2+2}{n^4+6} < \frac{n^2+2}{n^4}$$

How can we prove $\sum_{n=3}^{\infty} \frac{n^2+2}{n^4}$ converges?

$$\sum_{n=3}^{\infty} \frac{n^2+2}{n^4} = \sum_{n=3}^{\infty}\left[\frac{n^2}{n^4} + \frac{2}{n^4}\right] = \sum_{n=3}^{\infty} \frac{1}{n^2} + \sum_{n=3}^{\infty} \frac{2}{n^4}$$ where both series converge

p-series

Remember we can do the splitting only if we know for sure the splitted pieces converge.

Then we found the larger convergent series.

Then, by Comparison Test, the original series converges.

• **Ex:** compare...

→ If we had $\sum_{n=0}^{\infty} \frac{1}{3^n+n}$, we could find a larger convergent and show convergence.

→ But if we had $\sum_{n=0}^{\infty} \frac{1}{3^n-n}$, this wouldn't such an obvious case...

Limit Comparison Test:

Given $\sum a_n$ and $\sum b_n$ with $a_n > 0$ and $b_n > 0$, compute:

$$c_n = \lim_{n\to\infty} \frac{a_n}{b_n} \quad \text{or} \quad c_n = \lim_{n\to\infty} \frac{b_n}{a_n}.$$

If $0 < c < \infty$ (finite positive), then both series have the same convergence, i.e., both converge or both diverge.

• **Ex:** $\sum_{n=0}^{\infty} \frac{1}{3^n-n}$. We need a 2^{nd} series we know converges and "behaves" in about the same way.

Probably use $\sum_{n=0}^{\infty} \frac{1}{3^n}$ (which we know converges).

So, we have two options:

$$c = \lim_{n\to\infty} \frac{1}{3^n-n} \cdot \frac{3^n}{1} \quad \text{or} \quad c = \lim_{n\to\infty} \frac{1}{3^n} \cdot \frac{3^n-n}{1}$$

Which one is easier?

The second one:

$$c = \lim_{n\to\infty} \frac{3^n-n}{3^n} = \lim_{n\to\infty} \left(1 - \frac{n}{3^n}\right) = 1 - \lim_{n\to\infty}\left(\frac{n}{3^n}\right)^{\frac{\infty}{\infty}}$$

Then do L'hôpital

$$= 1 - \lim_{n\to\infty}\left(\frac{[n]'}{[3^n]'}\right) = 1 - \lim_{n\to\infty}\left(\frac{1}{3^n \cdot \ln(3)}\right)$$

$$= 1 - \frac{1}{\ln(3)}\left(\cancel{\lim_{n\to\infty} \frac{1}{3^n}}^{\,0}\right) = 1$$

Now, we have that $0 < c = 1 < \infty$. Then we know that both $\sum_{n=0}^{\infty} \frac{1}{3^n}$ and $\sum_{n=0}^{\infty} \frac{1}{3^n-n}$ have the same convergence.

We know $\sum_{n=0}^{\infty} \frac{1}{3^n}$ converges, then $\sum_{n=0}^{\infty} \frac{1}{3^n-n}$ converges too!

Ex:

- $\sum_{n=2}^{\infty} \frac{4n^2+n}{\sqrt[3]{n^7+n^3}}$ Observe: $\frac{4n^2+n}{\sqrt[3]{n^7+n^3}} \approx \frac{n^2}{n^{7/3}} = \frac{1}{n^{1/3}}$ which diverges (p-series test)

Comparison test won't give us a clear way out. Let's instead do Limit Comparison Test, and use $\frac{1}{n^{1/3}}$ as our second series.

$$C = \lim_{n\to\infty} \frac{4n^2+n}{\sqrt[3]{n^7+n^3}} \cdot \frac{n^{1/3}}{1} = \lim_{n\to\infty} \frac{4n^{7/3}+n^{4/3}}{\sqrt[3]{n^7+n^3}} = \frac{4}{\sqrt[3]{1}} = 4$$

Then $0 < C = 4 < \infty$. Then $\sum \frac{1}{n^{1/3}}$ and the original series have the same convergence. But we know $\sum \frac{1}{n^{1/3}}$ diverges, then our series $\sum_{n=2}^{\infty} \frac{4n^2+n}{\sqrt[3]{n^7+n^3}}$ diverges as well.