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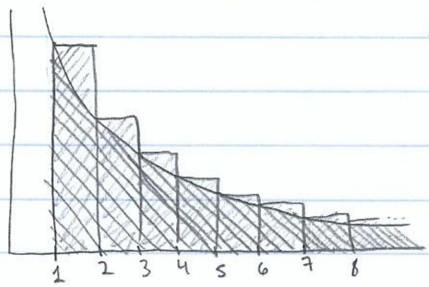
Convergence \ Divergence Tests:

INTEGRAL TEST:

Remember: Area under $f(x)=\frac{1}{x}$ on $[1,\infty)$

$\int_1^\infty \frac{1}{x}\,dx$ is the exact area

But, let's estimate (sum rectangles):



$$\text{Area} \approx \left(\tfrac{1}{1}\right)(1)+\left(\tfrac{1}{2}\right)(1)+\left(\tfrac{1}{3}\right)(1)+\left(\tfrac{1}{4}\right)(1)+\dots$$
$$=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots$$
$$=\sum_{n=1}^{\infty}\frac{1}{n}$$

So, the estimate of the area is the harmonic series.

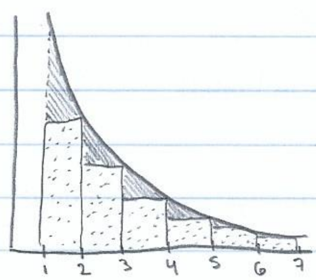
Note that your estimate is greater than the actual area.

$$\sum_{n=1}^{\infty}\frac{1}{n} > \int_1^\infty \frac{1}{x}\,dx = \infty$$

Then the harmonic series, if it has a value at all, is strictly bigger than ∞.

$\sum_{n=1}^{\infty}\frac{1}{n} > \infty$, then the harmonic series is divergent.

- Let's now do it with $\int_1^\infty \frac{1}{x^2}\,dx$ and $\sum_{n=1}^{\infty}\frac{1}{n^2}$.



$$\text{Area} \approx \frac{1}{2^2}(1)+\frac{1}{3^2}(1)+\frac{1}{4^2}(1)+\dots$$
$$=\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots=\sum_{n=2}^{\infty}\frac{1}{n^2}$$

The estimated area is less than the exact area $\int_1^\infty \frac{1}{x^2}\,dx$

$$\sum_{n=2}^{\infty}\frac{1}{n^2} < \int_1^\infty \frac{1}{x^2}\,dx = 1$$

Then we have that: $\sum_{n=2}^{\infty}\frac{1}{n^2} < 1$. Remember: $\sum_{n=1}^{\infty}\frac{1}{n^2} = 1+\sum_{n=2}^{\infty}\frac{1}{n^2}$

$$\downarrow$$

$$\left(\sum_{n=2}^{\infty}\frac{1}{n^2}\right)+1 < 1+1$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} < 2$$ and since all terms are positive, $\sum_{n=1}^{\infty}\frac{1}{n^2} > 0$

Then $0 < \sum_{n=1}^{\infty}\frac{1}{n^2} < 2$. Then this series converges.

- What's the integral test?

Given a series $\sum_{n=k}^{\infty} a_n$ where $a_n > 0$ and a_n must be decreasing for all n, (actually "eventually")
then find a function $f(x)$ so that $f(n) = a_n$. Then compute $\int_k^{\infty} f(x)dx$.

- If $\int_k^{\infty} f(x)dx$ converges, so does $\sum_{n=k}^{\infty} a_n$.
- If $\int_k^{\infty} f(x)dx$ diverges, so does $\sum_{n=k}^{\infty} a_n$.

CASES:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \underbrace{\sum_{n=3}^{\infty} a_n}_{\text{if } a_n \text{ now is all + and decreasing, then we can do INTEG. TEST!}}$$

$\downarrow$ let's say the first two terms are negative

Ex: Converge or Diverge?

- $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ → let's check first the necessary conditions.

$a_n = \frac{1}{n\ln(n)} > 0$ since $n, \ln(n) > 0$ for $n \geq 2$.

Now, since n and $\ln(n)$ increase always for $n \geq 2$,

then $\frac{1}{n\ln(n)}$ will always also decrease for $n \geq 2$.

Then we're ready to do the integral test.

$$\int_2^{\infty} \frac{1}{x\ln(x)}dx = \lim_{t\to\infty} \int_2^t \frac{1}{x\ln(x)}dx \qquad \begin{array}{l} u = \ln(x) \\ du = \frac{1}{x}dx \end{array}$$

$$= \lim_{t\to\infty} \int_2^t \frac{1}{u}du = \lim_{t\to\infty} \ln|\ln(x)| \Big|_2^t$$

$$= \lim_{t\to\infty} \left(\ln|\ln(t)| - \ln|\ln(2)| \right) = \infty$$

So the integral diverges. Then the series must also diverge by the Integral Test.