02-05-2019

EXAM 1 : NEXT THURSDAY (Feb. 14)

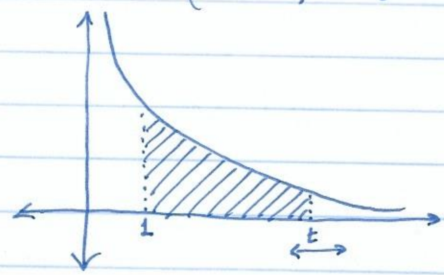
Improper Integrals

Ex:

Consider $\int_1^\infty (\ldots)\,dt$, and now $\int_1^t \frac{1}{x^2}dx = \frac{-1}{x}\Big|_1^t$. Notice, the second looks more feasible.

$\hookrightarrow \left(-\frac{1}{t}+1\right) \rightarrow \xrightarrow{\text{take: } t\to\infty} (1)$

Let's look at the graph:



So, this is how we're going to work out these integrals.

$$\int_1^\infty \frac{1}{x^2}dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}dx = \lim_{t\to\infty}\left[-\frac{1}{t}+1\right] = 1.$$

Ex:

$$\int_1^\infty \frac{1}{x}dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}dx = \lim_{t\to\infty}\left[\ln|x|\right]_1^t = \lim_{t\to\infty}\left(\ln|t| - \cancelto{0}{\ln|1|}\right)$$
$$= \lim_{t\to\infty}\ln|t| = +\infty$$

So, this integral doesn't converge to a value.

Let's remember:

$$\lim_{x\to\infty}\ln(x) = \infty \;,\; \lim_{x\to\infty}e^x = \infty \;,\; \lim_{x\to-\infty}e^x = 0$$

Also, depending on what the result ends up being, these integrals can be convergent (limits exist & are finite) or divergent ($\pm\infty$ or $\nexists$).

Fact:

$$\int_a^\infty \frac{1}{x^p}dx = \begin{cases} \text{convergent if } p>1 \\ \text{divergent if } p\le 1 \end{cases}$$

$a>1$

Ex:

$$\int_{-\infty}^0 \frac{1}{\sqrt{3-x}}dx = \lim_{t\to-\infty}\int_t^0 (3-x)^{-1/2}dx = \lim_{t\to-\infty}\left[-2(3-x)^{1/2}\right]_t^0$$
$$= \lim_{t\to-\infty}\left(-2(3)^{1/2} + 2(3-t)^{1/2}\right)$$
$$= -2\sqrt{3} + 2\lim_{t\to-\infty}\sqrt{\underset{-(-\infty)}{3-t}} = +\infty$$

It's divergent.

remember: $\lim_{t\to\infty}\int_{-t}^{t}(\dots)dx$ ~~$\otimes$~~ NO!

Ex: $\bullet \int_{-\infty}^{\infty} xe^{-x^2}dx$, We'll split the integral.

Quickly: $\int xe^{-x^2}dx = -\frac{1}{2}e^{-x^2}+C$

$\overset{?}{=} \int_{-\infty}^{0} xe^{-x^2}dx + \int_{0}^{\infty} xe^{-x^2}dx$

OK

Only valid if both of the new "integrals" converge.

Thus, if either of the new "integrals" diverges, then so does the original.

$= \lim_{t\to-\infty}\int_{t}^{0} xe^{-x^2}dx + \lim_{s\to\infty}\int_{0}^{s} xe^{-x^2}dx$

$= \lim_{t\to-\infty}\left[-\frac{1}{2}e^{-x^2}\right]_{t}^{0} + \lim_{s\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_{0}^{s}$

$= \lim_{t\to-\infty}\left[-\frac{1}{2}+\frac{1}{2}e^{-t^2}\right] + \lim_{s\to\infty}\left(-\frac{1}{2}e^{-s^2}+\frac{1}{2}\right)$

0 CONV 0 CONV

$= \left[-\frac{1}{2}\right]+\left[\frac{1}{2}\right]$

$= \emptyset$

Both convergent, then the orig. $\int_{-\infty}^{\infty}$ is also convergent and its value is Zero.

Let's go back to the case where the limits of the integral touch or go to zero and the function inside the integral seems to "break" at zero.

This could not necessarily happen at zero, but at also other "undefined" points of the function.