02-12-2019

## Arc length (cont'd):

$$L = \int ds$$

- $ds = \sqrt{1+(dy/dx)^2}\, dx$
- $ds = \sqrt{1+(dx/dy)^2}\, dy$

**Ex**: Find the length at

$y = \ln(\sec(x))$ on $0 \le x \le \frac{\pi}{4}$

$\hookrightarrow \frac{dy}{dx} = \frac{1}{\sec(x)} \sec x \tan x = \tan x$

$$ds = \sqrt{1+\tan^2 x}\, dx$$
$$= \sqrt{\sec^2 x}\, dx$$
$$= |\sec x|\, dx \text{, but positive on } 0 \le x \le \pi/4$$

Then: $= \int_0^{\pi/4} \sec x\, dx = \left[\ln|\sec x + \tan x|\right]_0^{\pi/4}$

$$= \left[\ln\left|\sec\left(\tfrac{\pi}{4}\right)+\tan\left(\tfrac{\pi}{4}\right)\right| - \ln|\sec(0)+\tan(0)|\right]$$

$$= \ln(\sqrt{2}+1)$$ Always make sure it's positive

**Ex**: Find length of

$x = \frac{2}{3}(y-1)^{3/2}$ for $1 \le y \le 4$ [second $ds$]

$\frac{dx}{dy} = \frac{2}{3}\cdot\frac{3}{2}(y-1)^{1/2}(1)$

$$ds = \sqrt{1+((y-1)^{1/2})^2}\, dy$$
$$= \sqrt{1+(y-1)}\, dy$$
$$= \sqrt{y}\, dy$$

Then:
$$L = \int_1^4 y^{1/2}\, dy$$
$$= \frac{2}{3} y^{3/2} \Big|_1^4$$
$$= \frac{2}{3}(8-1) = \frac{14}{3}$$

- Second solution: solving for $y$.

$\frac{3}{2}x = (y-1)^{3/2}$ limits: $x = \frac{2}{3}(1-1)^{3/2} \to 0$

$\left(\frac{3}{2}x\right)^{2/3} + 1 = y$ $\quad x = \frac{2}{3}(3)^{3/2} \to 2\sqrt{3}$

$\frac{dy}{dx} = \frac{2}{3}\left(\frac{3}{2}x\right)^{-1/3}\left(\frac{3}{2}\right)$

$$ds = \sqrt{1+\left(\tfrac{3}{2}x\right)^{-2/3}}\, dx$$
$$= \sqrt{1+\frac{1}{\left(\frac{3}{2}x\right)^{2/3}}}\, dx$$
$$= \sqrt{\frac{\left(\frac{3}{2}x\right)^{2/3}+1}{\left(\frac{3}{2}x\right)^{2/3}}}\, dx$$

$$\frac{\sqrt{\left(\frac{3}{2}x\right)^{2/3}+1}}{\sqrt{\left(\frac{3}{2}x\right)^{2/3}}} = \frac{\sqrt{\left(\frac{3}{2}x\right)^{2/3}+1}}{\left(\frac{3}{2}x\right)^{1/3}}$$
$$= \left(\tfrac{3}{2}x\right)^{-1/3}\sqrt{\left(\tfrac{3}{2}x\right)^{2/3}+1}\, dx$$

Thus: $\int_0^{2\sqrt{3}} \left(\frac{3}{2}x\right)^{-1/3}\sqrt{\left(\frac{3}{2}x\right)^{2/3}+1}\, dx$

$$= \int_0^{2\sqrt{3}} u^{1/2}\, du = \int_1^4 u^{1/2}\, du$$

$u = \left(\frac{3}{2}x\right)^{2/3}+1 \longrightarrow u = \left(\frac{3}{2}\cdot 0\right)^{2/3}+1 = 1$

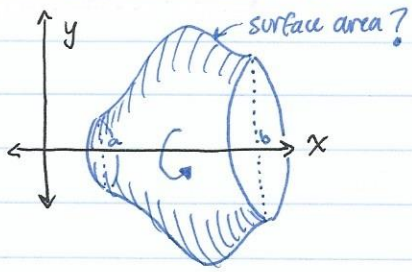
$du = \frac{2}{3}\left(\frac{3}{2}x\right)^{-1/3}\left(\frac{3}{2}\right)dx$ $\quad u = \left(\frac{3}{2}\cdot 2\sqrt{3}\right)^{2/3}+1$

$du = \left(\frac{3}{2}x\right)^{-1/3}dx$ $\quad \left(3\sqrt{3}\right)^{2/3}+1$

$\left(3^{3/2}\right)^{2/3}+1 = 4$

$= \int_1^4 u^{1/2}\, du = \dots = \frac{14}{3}$ [same answer]

## Surface Area of revolving bodies:




If we rotate around the $x$-axis:

$$\int 2\pi \cdot y \cdot ds$$

If we rotate around the $y$-axis:

$$\int 2\pi \cdot x \cdot ds$$

where $ds$ is one of the two we saw in the "Arc Length" section