01-28-2019

TRIG SUBSTITUTIONS (continued)

Ex: From last time.

- $\int \frac{\sqrt{25x^2-4}}{x}\,dx$. Let $x = \frac{2}{5}\sec\theta$

  So, $\sqrt{25x^2-4} = \dots = 2|\tan\theta|$.

  Here we'll assume positive because this is an indef. integral.

  $= 2\tan\theta$

Now, we'll proceed to plug in our new variable θ, remember to substitute all the terms in the integral $(x \to \theta)$.

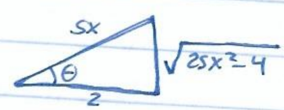
$$x = \frac{2}{5}\sec\theta \to dx = \frac{2}{5}\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{2\tan\theta}{\cancel{\left(\frac{2}{5}\right)}\cancel{\sec\theta}}\cancel{\left(\frac{2}{5}\right)}\cancel{\sec\theta}\tan\theta\,d\theta = \int 2\tan^2\theta\,d\theta$$

$$= 2\int \tan^2\theta\,d\theta = 2\int \sec^2\theta - 1\,d\theta = 2[\tan\theta - \theta] + C$$

Now, some work is need to have the answer in terms of x, not θ.

$$x = \frac{2}{5}\sec\theta \to \frac{5x}{2} = \frac{1}{\cos\theta} \to \cos\theta = \frac{2}{5x}$$



$$\theta = \cos^{-1}\left(\frac{2}{5x}\right) \qquad \tan\theta = \frac{\sqrt{25x^2-4}}{2}$$

$$= 2\left(\frac{\sqrt{25x^2-4}}{2} - \cos^{-1}\left(\frac{2}{5x}\right)\right) + C$$

$$= \sqrt{25x^2-4} - 2\cos^{-1}\left(\frac{2}{5x}\right) + C$$

Ex: Let's make it definite.

- $\int_{-4/5}^{-2/5} \frac{\sqrt{25x^2-4}}{x}\,dx$ where we'll let $x = \frac{2}{5}\sec\theta$, $dx = \frac{2}{5}\sec\theta\tan\theta\,d\theta$.

  $\sqrt{25x^2-4} = \dots = 2|\tan\theta|$. $\Rightarrow$ How to know + or $-$?

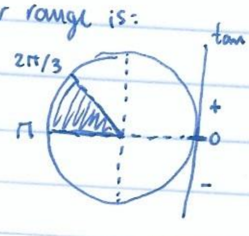
Let's convert the limits.

$$x = \frac{-4}{5} \to \frac{-4}{5} = \frac{2}{5}\sec\theta \to -2 = \sec\theta = \frac{1}{\cos\theta} \to \cos\theta = \frac{-1}{2} \Rightarrow \theta = \cos^{-1}\left(\frac{-1}{2}\right)$$

$$\theta = \frac{2\pi}{3}.$$

$$x = \frac{-2}{5} \to \frac{-2}{5} = \frac{2}{5}\sec\theta \to \sec\theta = -1 \to \cos\theta = -1 \to \theta = \pi.$$

our range is:



This means our $\tan\theta$ is negative. Then we get rid of abs. value by mult. "$\tan\theta$" by negative one.

Then, $\int_{-4/5}^{-2/5} \frac{\sqrt{25x^2-4}}{x}\,dx = \int_{\frac{2\pi}{3}}^{\pi} \frac{-2\tan\theta}{\cancel{\frac{2}{5}\sec\theta}} \cdot \cancel{\frac{2}{5}\sec\theta}\tan\theta\,d\theta$

$= -2\int_{2\pi/3}^{\pi} \sec^2\theta - 1\,d\theta$

$= -2\left[\tan\theta - \theta\right]\Big|_{2\pi/3}^{\pi}$

$= -2\left[(\tan\pi - \pi) - (\tan\frac{2\pi}{3} - \frac{2\pi}{3})\right]$

$= \frac{2\pi}{3} - 2\sqrt{3}$

Ex:

- $\int \frac{1}{x^4 \cdot \sqrt{9-x^2}}\,dx$. Let $x = 3\sec\theta$

Then $\sqrt{9-x^2} = \sqrt{9-9\sec^2\theta} = 3\sqrt{1-\sec^2\theta} = 3\sqrt{-\tan^2\theta} \rightarrow$ neg!! X

This substitution doesn't work then.

Let $x = 3\sin\theta$. [this way dx is positive, $dx = 3\cos\theta\,d\theta$]

$\sqrt{9-x^2} = \sqrt{9-(3\sin\theta)^2} = \sqrt{9-9\sin^2\theta} = 3\sqrt{1-\sin^2\theta} = 3\sqrt{\cos^2\theta} = 3|\cos\theta|$
$= 3\cos\theta$

Since it's an indef., we can drop abs. val. bars.

$= \int \frac{1}{3^4\sin^4\theta\,\cancel{(3\cos\theta)}}\,\cancel{3\cos\theta}\,d\theta = \int \frac{1}{3^4}\cdot\frac{1}{\sin^4\theta}\,d\theta$

$= \frac{1}{3^4}\int \frac{1}{\sin^4\theta}\,d\theta = \frac{1}{81}\int \csc^4\theta\,d\theta = \frac{1}{81}\int \csc^2\theta\,\csc^2\theta\,d\theta$

$u = \cot\theta$

$= \frac{1}{81}\int (\cot^2\theta + 1)\csc^2\theta\,d\theta \qquad du = -\csc^2\theta\,d\theta$

$= \frac{1}{81}\int (u^2+1)(-1)\,du$

$= \frac{-1}{81}\int u^2 + 1\,du = \frac{-1}{81}\left[\frac{1}{3}\cot^3\theta + \cot\theta\right] + C$