Integrals with trig. functions (cont'd)

Ex: \[ \int \sin^6(x) \cos^3(x) \, dx \]

\[ \sin^2 x + \cos^2 x = 1 \]

Rule of Thumb: look at the odd coefficient/exponent. One will go away
with "du" and the remaining even ones can be converted.

\[ u = \sin(x) \quad du = \cos(x) \, dx \quad dx = \frac{du}{\cos(x)} \]

\[ = \int \sin^6(x) (1 - \sin^2(x)) \cos(x) \, dx \]

\[ = \int u^6 (1 - u^2) \cos(x) \, du \]

\[ = \int u^6 - u^8 \, du = \frac{1}{7} u^7 - \frac{1}{9} u^9 + C \]

\[ = \frac{1}{7} \sin^7(x) - \frac{1}{9} \sin^9(x) + C \]

If both of your exponents are odd, you can pick either one, but one will be easier.
If both are even, then this technique doesn't work.

Ex:

- \[ \int \sin^2(2x) \cos^3(2x) \, dx \] There are 3 possible ways to work this out.

For this, we'll need to review some trig. formulas:

\[ \begin{align*}
(1) \quad \cos^2(x) &= \frac{1 + \cos(2x)}{2} \\
(2) \quad \sin^2(x) &= \frac{1 - \cos(2x)}{2} \\
(3) \quad \sin(2x) &= 2 \sin(x) \cos(x) \quad \rightarrow \text{we'll use this one}
\end{align*} \]

The following trick applies only when the power/exponent is the same for both

\[ = \int (\sin(2x) \cos(2x))^2 \, dx = \frac{1}{4} \int (\sin(2x))^2 \, dx \]

\[ = \frac{1}{4} \int \sin^2(2x) \, dx \quad \text{Well now use (2)} \]

\[ = \frac{1}{4} \int \left( \frac{1 - \cos(4x)}{2} \right) \, dx = \frac{1}{8} \int 1 - \cos(4x) \, dx = \frac{1}{8} \left( x - \frac{1}{4} \sin(4x) \right) + C \]

- \[ \int \sin^2(x) \cos^2(x) \, dx \] Let's now pick a different path.

\[ = \frac{1}{2} (1 - \cos(2x)) \frac{1}{2} (1 + \cos(2x)) \, dx = \frac{1}{4} \int 1 - \cos^2(2x) \, dx \]

Now use (4) or replace it with \( \sin^2(x) \)

\[ = \frac{1}{4} \int 1 - \frac{1}{2} (1 + \cos(2x)) \, dx \]

\[ = \frac{1}{4} \int 1 - \frac{1}{2} (1 + \cos(2x)) \, dx \]

And you solve it

Beware: Sometimes picking different paths will take you to different results.
Ex:
- \( \int \sec^3(x) \tan^5(x) \, dx \)

\[ u = \sec(x) \quad du = \sec(x) \tan(x) \, dx \quad dx = \frac{du}{\sec(x) \tan(x)} \]

\[ = \int \sec^8(x) \tan^4(x) \cdot \tan(x) \sec(x) \, dx \]

\[ = \int \sec^9(x) (\tan^2(x))^2 \tan(x) \sec(x) \, dx \]

\[ = \int \sec^9(x) (\sec^2(x) - 1)^2 \tan(x) \sec(x) \, dx \]

\[ = \int u^8 (u^2 - 1)^2 \, du = \int u^9 (u^4 + 1 - 2u^2) \, du = \int u^{12} + u^9 - 2u^6 \, du \]

\[ = \frac{1}{13} u^{13} + \frac{1}{10} u^{10} - \frac{2}{11} u^7 + C \]

\[ = \frac{1}{13} \sec^{13}(x) + \frac{1}{10} \sec^9(x) - \frac{2}{11} \sec^6(x) + C \]