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4.8 Differentials

Definition 4.8.1.

Let f be a function that is differentiable at c. The equation of the tangent line at the point $(c, f(c))$ is

$$y - f(c) = f'(c)(x-c)$$

$$y = f(c) + f'(c)(x-c)$$

and is called the tangent line approximation of f at c.

Note 4.8.2

Since c is a constant, y is a linear function
of x. Moreover, by restricting the values of x
to those sufficiently close to c, the values of
y can be used as approximate (to any desired
degree of accuracy) of the values of the
function f. In other words, as x approaches
c, the limit of y is $f(c)$

Example 4.8.3

Find the tangent line approximation of $f(x) = \sqrt{x}$ at $x = 16$. Then use it to approximate the number $\sqrt{16.5}$.

Derive $f'(x) = (\sqrt{x})' = (x^{1/2})'$

$$= \frac{1}{2} x^{-1/2}$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

find $f'(16)$. $f'(16) = \frac{1}{2\sqrt{16}} = \frac{1}{2(4)}$

$$f'(16) = \frac{1}{8}$$ Also $f(16) = \sqrt{16} = 4$

Substitute in $y = f(c) + f'(c)(x-c)$ for $c = 16$.

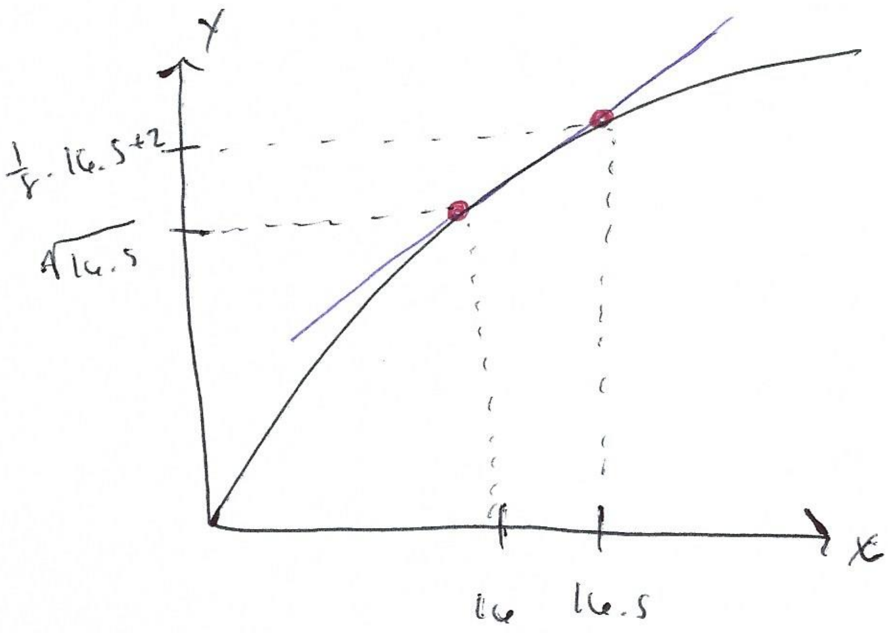
$$y = 4 + \frac{1}{8}(x - 16)$$

$$\boxed{y = \frac{1}{8}x + 2}$$

we now know $\sqrt{16.5} \approx \frac{1}{8} \cdot 16.5 + 2$

$$= \frac{1}{8} \cdot \frac{33}{2} + 2 = \frac{33}{16} + 2$$

$$= \frac{33 + 32}{16} = \frac{65}{16} \approx 4.0625$$



Definition 4.8.4

1. When the tangent line to the graph of f at the point $(c, f(c))$

$$\boxed{y = f(c) + f'(c)(x-c)}$$

is used as an approximation of the graph of x, the quantity $x-c$ is called the change of x, and is denoted by Δx. When Δx is small the change in Y (denoted by Δy) can be aproximated as shown.

$$\boxed{\begin{aligned}\Delta y &= f(c+\Delta x) - f(c)\\ &\approx f'(c)\,\Delta x\end{aligned}}$$

for such an approximation, the quantity Δx is traditionally denoted by dx.

2. Let $y = f(x)$ represent a function that is differentiable on an open interval containing x. The differential of x (denoted by dx) is any nonzero real number. The differential of y is

$$\boxed{dy = f'(x)\,dx}$$

Example 4.8.5

Let $y = \sqrt{x}$. Find dy when $x = 16$ and $dx = 0.5$. Compare this value with Δy for $x = 16$ and $\Delta x = 0.5$.

$$dy = f'(x)\,dx \quad , \quad f(x) = \sqrt{x}$$

$$= \frac{1}{2\sqrt{x}}\,dx \qquad x = 16,\ dx = 0.5$$

$$= \frac{1}{2\sqrt{16}}(0.5) = \frac{1}{16} \approx 0.0625$$

very close

$$\Delta y = f(c + \Delta x) - f(c) \quad , \quad c = x = 16 \ , \ \Delta x = 0.5$$

$$= f(16.5) - f(16)$$

$$= \sqrt{16.5} - \sqrt{16} \approx 4.0620 - 4$$

$$= 0.0620$$

Propagated error

$$\Delta y = f(x + \Delta x) - f(x)$$

$$\approx dy = f'(x)\,dx$$

Relative error

$$\frac{dy}{y} = \frac{f'(x)\,dx}{f(x)}$$

Example 4.8.4

The measured radius of a ball bearing is 0.7 inches. The measurement is correct to within 0.01 inches. Estimate the propagated error in the volume (v) of the ball bearing. Is the error large or small?

Volume of a ball (sphere) $V = \frac{4}{3}\pi r^3$, $r = 0.7$ inch

correct to 0.01 inches $\to$ $-0.01 \le \Delta r \le 0.01$

$$\Delta V = V(r + \Delta r) - V(r)$$
$$\approx dv$$
$$= V'(r)\,dr$$

If $V = \frac{4}{3}\pi r^3$, $V' = 4\pi r^2$.

So $V'(r)\,dr = 4\pi r^2 dr$. $dr = \pm 0.01$
(margin of error)

$= 4\pi r^2 (\pm 0.01)$, $r = 0.7$

$= 4\pi (0.7)^2 (\pm 0.01)$

Therefore $V'(r)dr = 4\pi(0.7)^2(0.01) \approx 0.06158$
and $-4\pi(0.7)^2(0.01) \approx -0.06158$
propagate error $= \Delta v \approx dv = v'(r)dr$

relative error

$$\frac{dv}{v} = \frac{\cancel{\frac{4}{3}\pi}\cdot 3\cancel{r^2}\,dr}{\cancel{\frac{4}{3}\pi}\,r^{\cancel{3}\,1}} = \frac{3}{r}dr = \frac{3}{0.7}(\pm 0.01)$$
$$= \pm 0.0429 = \pm 4.29\%$$

Small

Theorem 4.8.7

Let u and v be differential functions of x

1. Constant multiple: $d(cu) = c\,du$
2. Sum or difference: $d(u \pm v) = du \pm dv$
3. Product: $d(uv) = du \cdot v + u \cdot dv$
4. Quotient: $d\left(\frac{u}{v}\right) = \frac{du \cdot v - u \cdot dv}{v^2}$

The notation is called the Leibniz notation for differentials and derivatives.