Math 2413
Dr. Kennedy
4 Apr 2019
pg 57

Integrals

Ex $\displaystyle\int_0^{\pi/4} (\tan x)^{1/2} \sec^2 x \, dx = \int_0^1 u^{1/2}\, dx = \frac{2}{3} u^{3/2}\Big|_0^1 = \frac{2}{3}\left(1^{3/2} - 0^{3/2}\right) = \frac{2}{3}$

*careful! $\tan x$ and $\sec x$ have vert. asymptotes at $(2k+1)\frac{\pi}{2}$, k any integer

$u = \tan x$
$du = \sec^2 dx$
$u(0) = 0$
$u(\pi/4) = 1$

LOGARITHMS

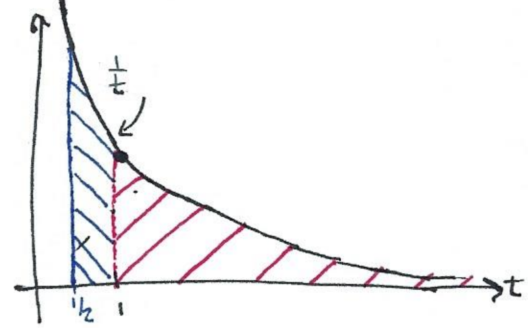
Def. A logarithm is a nonconstant differentiable function f defined on the positive real numbers such that for each $x>0, y>0$, $f(xy) = f(x) + f(y)$.

$\int \frac{1}{t}\, dt$ has an antiderivative for $t>0$, but if we use the power rule of integration, then $\int \frac{1}{t} dt = \int t^{-1} dt = \frac{t^0}{0} + C$, which is obviously a problem.

Define, for $x>0$, $\ln(x) = \displaystyle\int_1^x \frac{1}{t}\, dt$

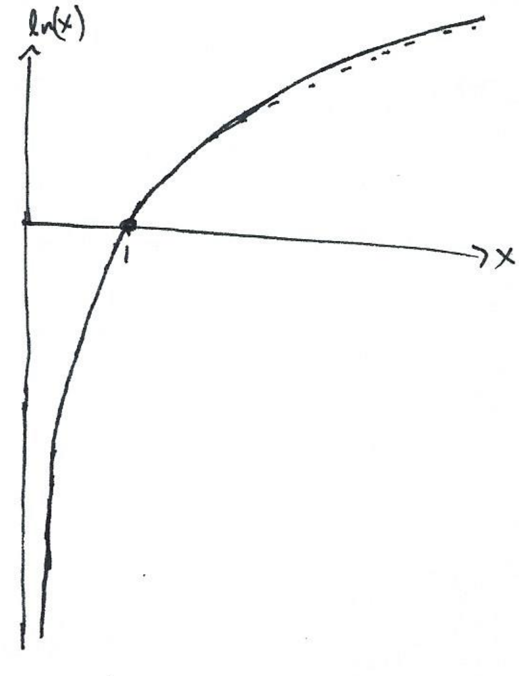
$\ln(1) = \displaystyle\int_1^1 \frac{1}{t} dt = 0$

$\ln(2) = \displaystyle\int_1^2 \frac{1}{t} dt$



Facts about $\ln(x)$

- $\ln(1) = 0$
- if $x>1$, $\ln(x)$ represents area, so $\ln(x) > 0$
- if $0<x<1$, $\ln(x) < 0$
- $D[\ln(x)] = \frac{1}{x}$ (by FTOC II)
- $D^2[\ln(x)] = -x^{-2} < 0$, so graph is concave down
- if $x,y>0$, $\ln(xy) = \ln(x) + \ln(y)$
  $\ln(x^r) = r\ln(x)$
- range of $\ln(x)$ is $(-\infty, \infty)$
  - neg. y-axis is vert. asymptote
  - no horiz. asymptote



LOGARITHMS

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Thm. If x, y positive, then $\ln(xy) = \ln(x) + \ln(y)$

Proof: $\ln(xy) = \int_1^{xy} \frac{1}{t}\,dt = \int_1^{x} \frac{1}{t}\,dt + \int_x^{xy} \frac{1}{t}\,dt = \ln(x) + \int_x^{xy} \frac{1}{t}\,dt$

* if $x>0$, $y>0$ then $xy>0$

NTS: $\int_x^{xy} \frac{1}{t}\,dt = \ln(y)$

Let $u = t/x$

then $du = \frac{dt}{x}$

and $t = ux$

$dt = x\,du$

* with respect to the integral, t is the variable, and x and xy are constants

$$\Longrightarrow \int_x^{xy} \frac{1}{t}\,dt = \int_{u(x)}^{u(xy)} \frac{1}{xu}\,x\,dx = \int_1^{y} \frac{1}{u}\,du = \ln(y)$$

So for $x, y > 0$, $\ln(xy) = \ln(x) + \ln(y)$.

Proposition if $x, y > 0$, then $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$

Proof: Consider $\ln\left(\frac{1}{y}\right)$.

$1 = (y)\left(\frac{1}{y}\right)$

$0 = \ln(1) = \ln\left(y \cdot \frac{1}{y}\right) = \ln(y) + \ln\left(\frac{1}{y}\right) \mapsto \ln(y) = -\ln\left(\frac{1}{y}\right)$

$\ln\left(\frac{x}{y}\right) = \ln\left(x \cdot \frac{1}{y}\right) = \ln(x) + \ln\left(\frac{1}{y}\right) = \ln(x) - \ln(y)$

Proposition if $\forall x > 0$, $r \in \mathbb{R}$, then $\ln(x^r) = r\ln(x)$

Proof: (show true for pos. integers)

(1) $\ln(x^1) = 1 \cdot \ln(x)$ ✓

(2) $\ln(x^2) = \ln(x \cdot x) = \ln(x) + \ln(x) = 2\ln(x)$ ✓

(3) $\ln(x^3) = \ln(x^2 \cdot x) = \ln(x^2) + \ln(x) = 2\ln(x) + \ln(x) = 3\ln(x)$ ✓

⋮

(n) $\ln(x^n) = n\ln(x^n)$