Theorem 5.3.11

If $f$ is integrable on the three closed intervals determined by $a$, $b$, and $c$, then

$$
\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx
$$

The diagram illustrates the sum of the areas under the curve of $f(x)$ from $a$ to $b$, split at $c$. The shaded area represents the integral from $a$ to $b$.
Example 5.3.12

1. Set up a definite integral that yields the area of the region (do not evaluate)

(a) \( g(y) = y^3 \)

2: \[ \int_0^2 y^3 \, dy \]
3: \( 2 \sin \left( \int_0^y 1 \, dy \right) \) 

Definite integrals are areas of the function.

So \( \int_0^2 y^3 \, dy = 2 \sin \int_0^y 1 \, dy \) is in terms of \( y \).

We can't use \( x \) sine the function doesn't border the \( x \)-axis.

(b) \( f(y) = (y - 2)^2 \)

2: \( f(y) \) borders the \( y \)-axis and \( x \)-axis, we can integrate in terms of either.

\( \sin \left( \int_0^y (y - 2)^2 \, dy \right) \) is in terms of \( y \) already,

\[ \int_0^2 (y - 2)^2 \, dy \]

For the integral in terms of \( x \),

Let \( f(y) = x = (y - 2)^2 \) and solve for \( y \).

\[ \pm \sqrt{x} = y - 2 \]

\[ 2 + \sqrt{x} = y \]

Notice the function is positive for all \( x \) in quadrant 1. Recall our graph is the opposite of this, so use \( -\sqrt{x} \).

So \( y = 2 + \sqrt{x} \)

\[ \int_0^4 \left( 2 - \sqrt{x} \right) \, dx \]
2. Evaluate $\int_{-1}^{1} |x| \, dx$ — Recall, integration is area.

So Area: $\int_{-1}^{1} |x| \, dx$

$$\text{Area} = \frac{(1)(1)}{2} + \frac{(1)(1)}{2} = \frac{1}{2} + \frac{1}{2} = 1$$

Area: $1 = \int_{-1}^{1} |x| \, dx$

$A = \frac{bh}{2}, \, b, h = 1$

**OR**

Let $\int_{-1}^{1} |x| \, dx = \int_{-1}^{0} |x| \, dx + \int_{0}^{1} |x| \, dx$

$$= \int_{-1}^{0} -x \, dx + \int_{0}^{1} x \, dx$$

$$= -\int_{-1}^{0} x \, dx + \int_{0}^{1} x \, dx$$

$$= \left[ -\frac{x^2}{2} \right]_{-1}^{0} + \left[ \frac{x^2}{2} \right]_{0}^{1} = \frac{1}{2} + \frac{1}{2} = 1$$

$$\int_{-1}^{1} |x| \, dx = \left[ -\frac{x^2}{2} \right]_{-1}^{0} + \left[ \frac{x^2}{2} \right]_{0}^{1} = \frac{1}{2} + \frac{1}{2} = 1$$

$$= \left. -\frac{x^2}{2} \right|_{-1}^{0} + \left. \frac{x^2}{2} \right|_{0}^{1} = \frac{1}{2} - \frac{1}{2} + \frac{1}{2} = \frac{1}{2}$$

$$= \int_{-1}^{1} |x| \, dx = \left. -\frac{x^2}{2} \right|_{-1}^{0} + \left. \frac{x^2}{2} \right|_{0}^{1} = \frac{1}{2} - \frac{1}{2} + \frac{1}{2}$$

$$= \frac{1}{2}$$
Theorem 5.4.1: The Fundamental Theorem of Calculus

If a function $f$ is continuous on the closed interval $[a, b]$ and $F$ is an antiderivative of $f$ on the interval $[a, b]$, then

$$\int_{a}^{b} f(x) \, dx = F(b) - F(a)$$

Note 5.4.2: Guidelines

1. Provided you can find the antiderivative of $f$, you can evaluate a definite integral without having to use the limit of a sum.

2. While applying the Fundamental Theorem of Calculus, the notation is as below

$$\int_{a}^{b} f(x) \, dx = \left[ F(x) \right]_{a}^{b} = F(b) - F(a)$$

3. It is not necessary to include the constant of integration $C$ in the antiderivative since

$$\int_{a}^{b} f(x) \, dx = \left[ F(x) + C \right]_{a}^{b}$$

$$= [F(b) + C] - [F(a) + C]$$

$$= F(b) - F(a) + C - C$$

$$= F(b) - F(a)$$
Example 5.4.3

1. Evaluate each definite integral

(a) \[ \int_0^2 |2x-1| \, dx \]

\[ \text{area} = \frac{1}{2} \cdot 1 + \frac{3}{2} \cdot 3 \]

\[ = \frac{1}{2} + \frac{9}{2} = \frac{5}{2} \]

Rewrite integral

\[ = \int_0^{\frac{3}{2}} (2x-1) \, dx + \int_{\frac{3}{2}}^{2} (2x-1) \, dx \]

we apply a

neg. sign to this

is the left side of

the abs value

\[ = -\left[ x^2 - x \right]_0^{\frac{3}{2}} + \left[ x^2 - x \right]_{\frac{3}{2}}^2 \]

\[ = \left\{ \left[ (\frac{3}{2})^2 - \frac{3}{2} \right] - \left[ 0^2 - 0 \right] \right\} + \left\{ \left[ (2)^2 - (2) \right] - \left[ (\frac{3}{2})^2 - (\frac{3}{2}) \right] \right\} \]

\[ = -\left[ \frac{1}{4} - \frac{3}{2} \right] + \left[ 2 - \left[ \frac{1}{4} - \frac{3}{2} \right] \right] \]

\[ = -\frac{1}{4} + \frac{1}{2} + 2 - \frac{1}{4} + \frac{1}{2} \]

\[ = \frac{3}{2} + 1 = \frac{5}{2} \]
2. Find the area of the region bounded by the graph of \( y = \frac{1}{x} \), the \( y \)-axis, and the vertical lines \( x = 1 \) and \( x = e \).

So \( \int_1^e \frac{1}{x} \, dx \) will find the area.

\[
\int_1^e \frac{1}{x} \, dx = \left[ \ln|x| \right]_1^e = \ln e - \ln 1 = 1 - 0 = 1
\]

\textbf{Note:} If you integrate \( \ln|x| \) and get \( \ln x \), we write \( \ln |x| \) since \( x \geq 0 \) for \( x \).
Theorem 5.4.4: Mean Value Theorem for Integrals

If \( f \) is continuous on the closed interval \([a, b]\), then there exists a number \( c \) in the closed interval \([a, b]\) such that

\[
\int_a^b f(x) \, dx = f(c) (b-a)
\]

Definition 5.4.4:
If \( f \) is integrable on the closed interval \([a, b]\), then the average value of \( f \) is

\[
\frac{1}{b-a} \int_a^b f(x) \, dx
\]

average value
Theorem 5.4.8: The Second Fundamental Theorem of Calculus

If $f$ is continuous on an open interval $I$ containing $a$, then for all $x$ in the interval

$$\frac{d}{dx} \left[ \int_a^x f(t) \, dt \right] = f(x)$$

In general, we have

$$\frac{d}{dx} \left[ \int_{g(x)}^{f(x)} h(t) \, dt \right] = h(f(x))f'(x) - h(g(x))g'(x)$$

Example 5.4.9

1. Evaluate $\frac{d}{dx} \int_{-x}^{3x+2} \sqrt{t^2+1} \, dt$

$$= \int (3x+2)(3x+2)' - f(-x)(-x)'$$

$$= \sqrt{(3x+2)^2+1} \cdot 3 - \sqrt{(-x)^2+1} \cdot (-1)$$