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Definition 8.7.9. If we have a limit of the form

$$\lim_{x\to a} f(x) - g(x)$$

where $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then the limit may/may not exist and is in indeterminate form of type $\infty - \infty$.

Note 8.7.10 we can handle these types of limits by using factoring, rationalizing, and combining terms to rewrite the difference as a quotient. Then we will have the form $\frac{0}{0}$, or $\frac{\pm\infty}{\infty}$ to use L'Hôpitals Rule.

Example 8.7.11 Evaluate the limit

1. $\lim_{x\to(\pi/2)^-} (\sec x - \tan x)$ - direct sub gives $\infty - \infty$ & convert the difference.

$= \lim_{x\to(\pi/2)^-} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right)$ & rewrite as fractions and combine. $\sec x = \frac{1}{\cos x}$, $\tan x = \frac{\sin x}{\cos x}$

$= \lim_{x\to(\pi/2)^-} \left(\frac{1-\sin x}{\cos x}\right)$ - direct sub gives $\frac{0}{0}$ use L'Hôpital's Rule

$= \lim_{x\to(\pi/2)^-} \frac{-\cos x}{-\sin x} = \lim_{x\to(\pi/2)^-} \frac{\cos x}{\sin x} = \frac{0}{1}$

$= \boxed{0}$

2. $\lim_{x \to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right)$ → direct sub gives $\infty - \infty$

Rewrite

$= \lim_{x \to 1^+} \left(\frac{1}{\ln x} \cdot \frac{x-1}{x-1} - \frac{1}{x-1} \cdot \frac{\ln x}{\ln x} \right)$ → find a common denominator

$= \lim_{x \to 1^+} \left(\frac{x-1}{(\ln x)(x-1)} - \frac{\ln x}{(\ln x)(x-1)} \right)$

$= \lim_{x \to 1^+} \left(\frac{x - 1 - \ln x}{(\ln x)(x-1)} \right)$ → direct sub gives $\frac{0}{0}$ Use L'Hôpitals

$= \lim_{x \to 1^+} \frac{1 - \frac{1}{x}}{\ln x + 1 - \frac{1}{x}}$ → since $(\ln x)(x-1) = x\ln x - \ln x$ and $D(x\ln x - \ln x) = \ln x + 1 - \frac{1}{x}$ where $D(x\ln x) = \ln x + 1$ (by product rule) and $D(-\ln x) = -\frac{1}{x}$

$= \frac{0}{0}$ Use L'Hôpitals again

$= \lim_{x \to 1^+} \frac{\frac{1}{x^2}}{\frac{1}{x} + \frac{1}{x^2}} = \frac{1}{1+1} = \boxed{\frac{1}{2}}$

$D(1 - \frac{1}{x}) = D(1 - (x)^{-1})$
$= -- (x)^{-2} = x^{-2}$ → since $x^{-1} = \frac{1}{x}$, we can change the fraction to use the general power rule.
$= \frac{1}{x^2}$

Note 8.7.12 Several indeterminate forms arise from

$$\lim_{x \to a} (f(x))^{g(x)}$$

1. $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ → we get type 0^0
2. $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = 0$ → we get type ∞^0
3. $\lim_{x \to a} f(x) = 1$ and $\lim_{x \to a} g(x) = \pm\infty$ → we get type 1^∞

Each case can be handled by either taking the natural log such that

$$\text{if } y = \lim_{x \to a} (f(x))^{g(x)}, \text{ then } \ln y = \lim_{x \to a} g(x) \ln f(x)$$

or by taking the exponential

$$\lim_{x \to a} (f(x))^{g(x)} = \lim e^{g(x) \ln f(x)}$$

Either form yields $\lim_{x \to a} g(x) \ln f(x)$ which is the indeterminate form of type $0 \cdot \infty$ (which we know how to handle)

Example 8.7.13 Calculate the limit.

1. $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$ → direct sub → 1^∞ → indeterminate form

Let $y = \lim_{x \to \infty} \left(1+\frac{1}{x}\right)^x$. Then $\ln y = \ln \lim_{x \to \infty} \left(1+\frac{1}{x}\right)^x$

$$= \lim_{x \to \infty} x \ln\left(1+\frac{1}{x}\right)$$

Natural log allows us to bring down the exponent

So $\ln y = \lim_{x \to \infty} x \ln\left(1+\frac{1}{x}\right)$ → direct sub → $\infty \cdot 0$ form

Rewrite to get the other form ($\frac{0}{0}$ or $\frac{\infty}{\infty}$)

$$\ln y = \lim_{x \to \infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$$

– direct sub gives $\frac{0}{0}$ → use L'Hôpitals Rule!

$$\ln y = \lim_{x \to \infty} \frac{\frac{1}{1+\frac{1}{x}} \cdot \cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{-\frac{1}{x^2}}}$$

Since $D\left(\ln\left(1+\frac{1}{x}\right)\right) = \frac{1}{1+\frac{1}{x}} \cdot \left(-\frac{1}{x^2}\right)$

$D\left(\frac{1}{x}\right) = D\left((x)^{-1}\right) = -1 \cdot x^{-2} = -\frac{1}{x^2}$

Recall $D(\ln u) = \frac{1}{u} \cdot u'$

$$= \lim_{x \to \infty} \frac{1}{1+\frac{1}{x}}$$ – direct sub

$\ln y = 1$. Since $\ln y = 1$,

$$\cancel{e}^{\cancel{\ln} y} = e^1$$

raise the exponential $e^{\ln} = 1$

$$\boxed{y = e}$$

So if $y = \lim_{x \to \infty} \left(1+\frac{1}{x}\right)^x$ and $y = e$,

$$\boxed{\lim_{x \to \infty} \left(1+\frac{1}{x}\right)^x = e}$$

2. $\lim_{x \to 0^+} (\sin x)^x$ → Direct sub gives 0^0 → rewrite

let $y = \lim_{x \to 0^+} (\sin x)^x$ and $\ln y = \lim_{x \to 0^+} \ln(\sin x)^x$

$\ln y = \lim_{x \to 0^+} \ln(\sin x)^x$ ln allows us to bring down the exponent

$= \lim_{x \to 0^+} x \ln(\sin x)$ → Direct sub gives $0 \cdot (-\infty)$ → rewrite

$= \lim_{x \to 0^+} \dfrac{\ln(\sin x)}{\frac{1}{x}}$ → direct sub gives $-\frac{\infty}{\infty}$ use L'Hôpitals

$= \lim_{x \to 0^+} \dfrac{\frac{1}{\sin x} \cdot \cos x}{-\frac{1}{x^2}}$

$\frac{1}{x} = (x)^{-1}$, so $D(\frac{1}{x}) = D(x^{-1})$
$= -x^{-2}$
$= -1/x^2$

$= -\lim_{x \to 0^+} \dfrac{x^2 \cos x}{\sin x}$ → direct sub gives $\frac{0}{0}$ use L'Hôpitals

$= -\lim_{x \to 0^+} \dfrac{2x(\cos x) + x^2(-\sin x)}{\cos x}$

$D(x^2 \cos x) = 2x \cos x + x^2(-\sin x)$ via product rule

Direct sub

$= \dfrac{0}{1} = \boxed{0}$

So $\ln y = 0$ → solve for y

$e^{\ln y} = e^0$

$y = e^0$, $\boxed{y = 1}$ → $y = \lim_{x \to 0^+} (\sin x)^x$

$\boxed{y = \lim_{x \to 0^+} (\sin x)^x = 1}$