April 11, 2019

5.2 Area

Definition 5.2.1.

The sum of n terms $a_1, a_2, \ldots, a_n$ is written

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

where i is the index of summation, a_i is the ith term of the sum and the upper and lower bounds of summation are n and 1.

Example 5.2.2. Examples of sigma notation

1. $\sum_{k=1}^{4} k = 1+2+3+4$

2. $\sum_{i=2}^{4} \left(\frac{1}{\sqrt{i}} - 3\right)^2 = \left(\frac{1}{\sqrt{2}} - 3\right)^2 + \left(\frac{1}{\sqrt{3}} - 3\right)^2 + \left(\frac{1}{\sqrt{4}} - 3\right)^2$

3. $\sum_{i=1}^{n} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$

Theorem 5.2.3. Properties of summation

1. $\sum_{i=1}^{n} k a_i = k \sum_{i=1}^{n} a_i$

2. $\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

3. $\sum_{i=1}^{n} c = cn$, where c is a constant

4. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

5. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

6. $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Example 5.2.4.

1. Use Sigma notation to write the sum.

a) $\frac{9}{1+1} + \underbrace{\frac{9}{1+2}}_{+1} + \cdots + \frac{9}{1+14}$

we start at 1
and end at 14

end at $i=14$ $\rightarrow$

$$\sum_{i=1}^{14} \frac{9}{1+i} = \frac{9}{1+1} + \frac{9}{1+2} + \cdots + \frac{9}{1+14}$$

$\uparrow$
Start at $i=1$

(171)

b) $\left(\left(\frac{2}{n}\right)^3 - \frac{2}{n}\right)\left(\frac{2}{n}\right) + \cdots + \left(\left(\frac{2n}{n}\right)^3 - \frac{2n}{n}\right)\left(\frac{2}{n}\right)$

$$= \sum_{i=1}^{n} \left(\left(\frac{2i}{n}\right)^3 - \frac{2i}{n}\right)\left(\frac{2}{n}\right)$$ ← we consistently keep $\frac{2}{n}$.

The first term shows

$$\left(\left(\frac{2(1)}{n}\right) - \frac{2(1)}{n}\right)\left(\frac{2}{n}\right)$$

So we start at $i=1$ and end at $i=n$

Last term

$$\left(\left(\frac{2n}{n}\right)^3 - \frac{2n}{n}\right)\left(\frac{2}{n}\right)$$

2. Evaluate the sum $\sum_{i=1}^{n} \frac{i+1}{n^2}$ for $n=10$ and $n=100$.

$\underline{n=10}$

$$\sum_{i=1}^{10} \frac{i+1}{10^2} = \frac{1}{10^2}\sum_{i=1}^{10} i+1$$ since $\frac{1}{10}$ is a constant

$$= \frac{1}{10^2}\left[\sum_{i=1}^{10} i + \sum_{i=1}^{10} 1\right]$$ since $\sum_{i=1}^{n} a_i \pm b_i = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

$$= \frac{1}{10^2}\left[\sum_{i=1}^{10} i + 10\right]$$ since $\sum_{i=1}^{n} c = cn$

$$= \frac{1}{10^2}\left[\frac{10(11)}{2} + 10\right]$$ since $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$$= \frac{1}{10^2}\left[\frac{110}{2} + 10\right]$$

$$= \frac{1}{100}[55+10] = \frac{1}{100}[65) = \boxed{\frac{13}{10}}$$

$\underline{n=100}$

$$\sum_{i=1}^{100} \frac{i+1}{100^2} = \frac{1}{100^2}\sum_{i=1}^{100} i+1 = \frac{1}{100^2}\left[\sum_{i=1}^{100} i + \sum_{i=1}^{100} 1\right]$$

$$= \frac{1}{100^2}\left[\frac{100(101)}{2} + 100\right] = \boxed{\frac{103}{2}}$$

Evaluate

$$\sum_{k=1}^{10} \frac{6k(k-1)}{10^3} = \frac{6}{10^3}\sum_{k=1}^{10} k(k-1)$$ $\frac{6}{10^3}$ is a constant multiple

$$= \frac{6}{10^3}\sum_{k=1}^{10} k^2 - k$$ expand $k(k-1) = k^2 - k$

$$= \frac{6}{10^3}\left[\sum_{k=1}^{10} k^2 - \sum_{k=1}^{10} k\right]$$ separate sum/difference

$$= \frac{6}{10^3}\left[\frac{10(11)(21)}{6} - \frac{10(11)}{2}\right]$$

Recall

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

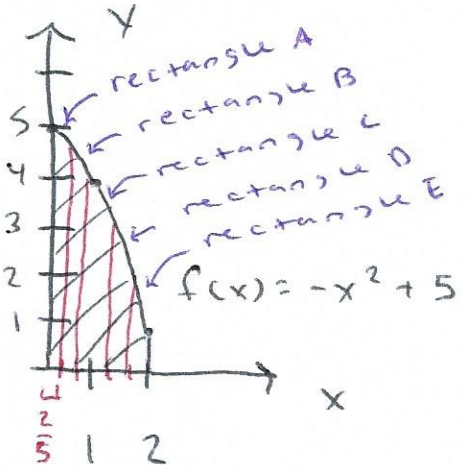
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

can simplify

$$= \frac{6}{1000}\left[385 - 55\right]$$

$$= \frac{6}{1000}\left[330\right] = \frac{1980}{1000} = \frac{198}{100} = \frac{99}{50}$$

4. Use left and right endpoints and 5 rectangles to find 2 approximations of the area of the region between the graph of $f(x) = -x^2 + 5$ and the x-axis over $[0, 2]$. Sketch the region.



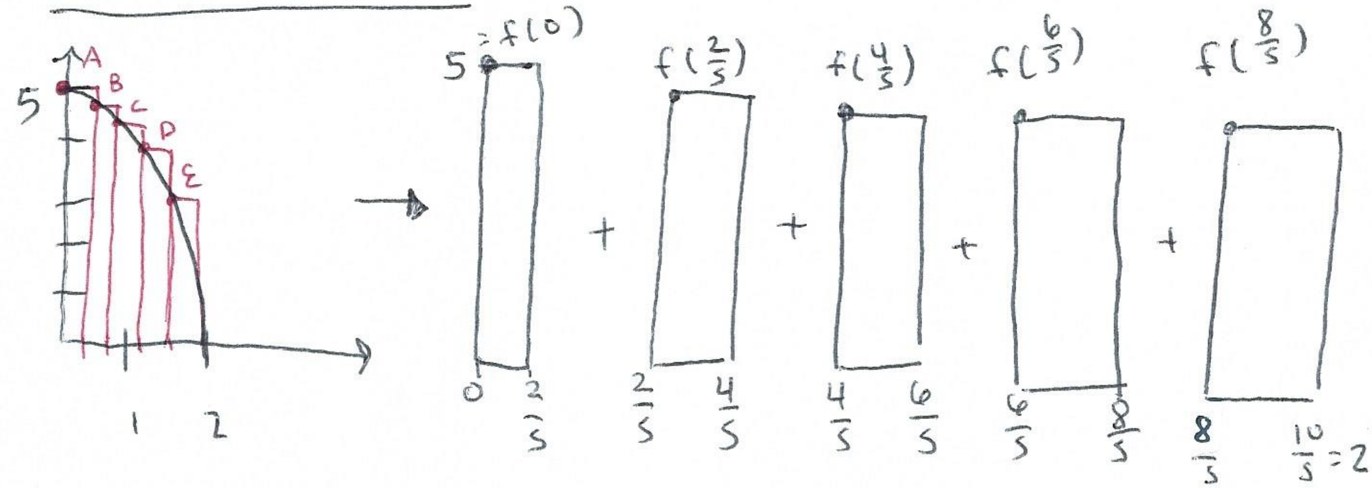
Find 5 rectangles

We want them equally spaced throughout $[0, 2]$.

$[0, 2]$ is 2 units.

2 units in 5 equal parts is $\frac{2}{5}$ units per rectangle.

Left endpoints



A + B + C + D + E

calculate areas (A = hw (height · width)) Notice each rectangle has the same width

$$A = A_A + A_B + A_C + A_D + A_E$$

$$= \left(\tfrac{2}{5}\right)(5) + \left(\tfrac{2}{5}\right)f\left(\tfrac{2}{5}\right) + \left(\tfrac{2}{5}\right)f\left(\tfrac{4}{5}\right) + \left(\tfrac{2}{5}\right)f\left(\tfrac{6}{5}\right) + \left(\tfrac{2}{5}\right)f\left(\tfrac{8}{5}\right)$$

$$= \tfrac{2}{5}\left(5 + f\left(\tfrac{2}{5}\right) + f\left(\tfrac{4}{5}\right) + f\left(\tfrac{6}{5}\right) + \tfrac{8}{5}\right)$$

$$= \tfrac{2}{5} \sum_{i=1}^{5} f(x_i), \text{ where } x_i = a + \Delta x(i-1)$$