CHAPTER 5: Integration

5.1 Antiderivatives and Indefinite Integration

Definition 5.1.1. A function $F$ is an antiderivative of $f$ on an interval $I$ when $F'(x) = f(x)$ for all $x$ in $I$.

Example 5.1.2. Find some antiderivatives of $f(x) = 2$.

$$F'(x) = f(x) = 2.$$  
$$F'(x) = 2x + c,$$  $c$ is a constant

Theorem 5.1.3. If $F$ is an antiderivative of $f$ on an interval $I$, then the most general antiderivative of $f$ on $I$ is

$$F(x) + C,$$  where $C$ is an arbitrary constant
Now we have the integral (indefinite form)

\[ Y = \int f(x) \, dx = F(x) + C \]

where

- \( f(x) \): integrand
- \( dx \): variable of integration
- \( F(x) \): antiderivative of \( f(x) \)
- \( C \): an arbitrary constant

**Note 5.1.5** The inverse nature of integration and differentiation can be verified by substituting \( F'(x) \) for \( f(x) \) in the indefinite integration defined to obtain

\[ \int F'(x) \, dx = F(x) + C \]

Moreover, if \( \int f(x) \, dx = F(x) + C \), then

\[ \frac{d}{dx} \left( \int f(x) \, dx \right) = f(x) \]
Basic Integration Rules

Differentiation Formula | Integration Formula
\[ \frac{d}{dx} [c] = 0 \quad \rightarrow \quad \int 0 \, dx = c \]
\[ \frac{d}{dx} [kx] = k \quad \rightarrow \quad \int k \, dx = kx + c \]
\[ \frac{d}{dx} [kf(x)] = kf'(x) \quad \rightarrow \quad \int kf(x) \, dx = k \int f(x) \, dx \]
\[ \frac{d}{dx} [f(x) \pm g(x)] = f'(x) \pm g'(x) \quad \rightarrow \quad \int [f(x) \pm g(x)] \, dx = \int f(x) \, dx \pm \int g(x) \, dx \]
\[ \frac{d}{dx} [x^n] = nx^{n-1} \quad \rightarrow \quad \int x^n \, dx = \frac{x^{n+1}}{n+1} + c, \; n \neq -1 \]
\[ \frac{d}{dx} [\sin x] = \cos x \quad \rightarrow \quad \int \cos x \, dx = \sin x + c \]
\[ \frac{d}{dx} [\cos x] = -\sin x \quad \rightarrow \quad \int \sin x \, dx = -\cos x + c \]
\[ \frac{d}{dx} [e^x] = e^x \quad \rightarrow \quad \int e^x \, dx = e^x + c \]
\[ \frac{d}{dx} [a^x] = (\ln a) a^x \quad \rightarrow \quad \int a^x \, dx = \left(\frac{1}{\ln a}\right) a^x + c \]
\[ \frac{d}{dx} [\ln x] = \frac{1}{x}, \; x > 0 \quad \rightarrow \quad \int \frac{1}{x} \, dx = \ln |x| + c \]
Example 5.1.6 Find the indefinite integral

1. \[ \int 5x^2 + \frac{3}{x} - \frac{1}{3\sqrt{x^2}} \, dx \]

\[ = \int 5x^2 \, dx + \int \frac{3}{x} \, dx - \int \frac{1}{3\sqrt{x^2}} \, dx \]

\[ = 5 \int x^2 \, dx + 3 \int \frac{1}{x} \, dx - \int \frac{1}{3\sqrt{x^2}} \, dx \]

\[ = 5 \left[ \frac{x^3}{3} + C_1 \right] + 3 \ln |x| + C_2 - \int \frac{1}{3\sqrt{x^2}} \, dx \]

Recall, \( \int \frac{1}{x} \, dx = \ln |x| + C \)

\[ = \frac{5x^3}{3} + C_1 + 3 \ln |x| + 3C_2 - \int \frac{1}{3\sqrt{x^2}} \, dx \]

\[ = \frac{5x^3}{3} + C_1 + 3 \ln |x| + 3C_2 - \left[ \frac{x^{-2/3} + 1}{2} + C_3 \right] \]

\[ = \frac{5x^3}{3} + C_1 + 3 \ln |x| + 3C_2 - \frac{3}{3\sqrt{x}} - C_3 \]

Rewrite all constants as one constant

\[ = \frac{5x^3}{3} + 3 \ln |x| - 3\sqrt[3]{x} + C \]

\[ \text{where } C = C_1 + 3C_2 - C_3, \]

\[ \text{an arbitrary constant} \]

Check, \[ \left[ \frac{5x^3}{3} + 3 \ln |x| - 3\sqrt[3]{x} + C \right]' = \int 5x^2 + \frac{3}{x} - \frac{1}{3\sqrt{x^2}} \, dx \]

\[ = 5x + \frac{3}{x} - \frac{1}{3\sqrt{x^2}} \] \(\checkmark\)
2. \[ \int \frac{x+1}{\sqrt{x}} \, dx \]

\[ = \int (x+1)(x^{-1/2}) \, dx \]

\[ = \int x^{1/2} - x^{-1/2} \, dx \]

\[ = \int x^{1/2} \, dx + \int x^{-1/2} \, dx \]

\[ = \frac{x^{(1/2+1)}}{1/2 + 1} + C_1 + \frac{x^{(-1/2+1)}}{-1/2 + 1} + C_2 \]

\[ = \frac{x^{3/2}}{3/2} + C_1 + \frac{x^{1/2}}{1/2} + C_2 \]

\[ = \frac{2x^{3/2}}{2} + C_1 + 2\sqrt{x} + C_2 \]

\[ = \frac{2}{3} (x^{3/2}) + 2\sqrt{x} + C \]
3. \[ \int \frac{\sin x}{\cos^2 x} \, dx \]
\[ = \int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} \, dx \]
\[ = \int \tan x \cdot \sec x \, dx \]
\[ = \sec x + C \]

4. \[ \int (x^2+1)^2 + 3\sqrt{t}(t-4) \, dt \]

Rewrite/Expand

\[ = \int (x^4+2x^2+1 + 3\sqrt{t} \cdot (t - 3\sqrt{t} \cdot 4 \, d \sqrt{t}) \]

Simplify

\[ = \int (x^4+2x^2+1 + (4^{1/3} - 4^{1/3} + 1) \, d \sqrt{t}) \]

Integrate

\[ = \frac{(x+1)^{2+1}}{4+1} + \frac{2^{2+1}}{2+1} + t + \left( 4^{1/3} - 4^{1/3} + 1 \right) \]

\[ = \frac{5^5}{5} + \frac{2^{4/3}}{3} + t + \frac{3^{7/3}}{7} - 3^{4/3} + C \]
Definition 5.1.7.

We need additional information to find particular solutions. Given \( y = f(x) \) for one value of \( x \) is an initial condition that will grant such a solution.

Example 5.1.8

1. Find the particular solution that satisfies the differential equation and the initial condition.

   (a) \( f'(x) = e^x \), \( f(0) = 3 \)

   Initial Condition

   Find \( f(x) \).

   \[ f(x) = \int f'(x) \, dx \]
   \[ = \int e^x \, dx = e^x + c \]

   Find \( c \), when \( f(0) = 3 \)

   \[ f(x) = e^x + c \]
   \[ f(0) = 3 = e^0 + c \]
   \[ 3 = 1 + c \]
   \[ c = 2 \]

   Substitute \( c \) into \( f(x) \)

   Since \( f(x) = e^x + c \) and \( c = 2 \) for \( f(0) = 3 \),

   \[ f(x) = e^x + 2 \]
2. A ball is thrown upward with an initial velocity of 64 feet per second from an initial height of 80 feet. Use \( a(t) = -32 \) ft/s^2 as the acceleration due to gravity.

(a) Find the position function giving the height \( s \) as a function of time \( t \).

Recall, \( s'(t) = v(t) \)
\( s''(t) = v'(t) = a(t) \)

So \( v(t) = \int a(t) \, dt \)
\[ = \int -32 \, dt \]
\[ v(t) = -32t + c \]

Since initial velocity \( v(t=0) = 64 \) ft/s,
\[ v(0) = 64 = -32(0) + c \]
\[ c = 64 \]

Then \( v(t) = -32t + 64 \).

Find \( s(t) \)
\[ s(t) = \int v(t) \, dt \]
\[ = \int -32t + 64 \, dt \]
\[ = -16t^2 + 64t + c \]

Since initial height \( s(t=0) = 80 \) feet,
\[ s(0) = 80 \]
\[ 80 = -16(0)^2 + 64(0) + c \]
\[ c = 80 \]

Then \( s(t) = -16t^2 + 64t + 80 \).
Example 5.1.2.
2. cont

(b) When does the ball hit the ground?
In other words, at what time \( t \), is the height \( s \) equal to zero (ground)?

Since \( s(t) = -16t^2 + 64t + 80 \), \( s(t) = 0 \) is when the ball hits the ground.

\[
0 = -16t^2 + 64t + 80 \\
0 = -16(t^2 - 4t - 5) \\
0 = -16(t - 5)(t + 1)
\]

\( t = 5 \) or \( t = -1 \)

We can't have negative time.

So after \( t = 5 \) seconds, the ball hits the ground.