Fundamental Theorem of Calculus

Math 2413
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pg 53

Suppose f is continuous on $[a,b]$ or has finitely many jump discontinuities, then:

FTOC I 1) $\int_a^b f(x)\,dx = F(b) - F(a)$, where $F'(x) = f(x)$ for all $x \in [a,b]$.

FTOC II 2) The function $g(x) = \int_a^x f(t)\,dt$, $a \le x \le b$, is continuous on $[a,b]$,

differentiable on (a,b), and $g'(x) = f(x)$ for all $x \in (a,b)$.

Ex. $\int_0^{\pi/4} \sec(\theta)\tan(\theta)\,d\theta = \sec(\theta)\Big|_0^{\pi/4} = \sec(\frac{\pi}{4}) - \sec(0) = \sqrt{2} - 1$

Ex. What is wrong?

$\int_{-2}^{1} x^{-4}\,dx = -\frac{x^{-3}}{3}\Big|_{-2}^{1} = -\frac{1}{3} + \frac{(-2)^{-3}}{3} = -\frac{1}{3} - \frac{1}{8\cdot 3} = \frac{-9}{24} = \frac{-3}{8}$ * x^{-4} is always non-negative, so this is a red flag.

Since x^{-4} is not continuous on $[-2,1]$, the first part of FTOC (f cont on $[a,b]$) is not satisfied.

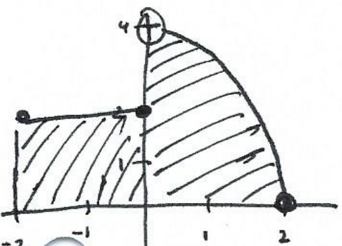
Ex. $\int_0^1 [3 + x\sqrt{x}]\,dx = \int_0^1 3\,dx + \int_0^1 x\sqrt{x}\,dx = 3\int_0^1 dx + \int_0^1 x^{3/2}\,dx = 3x\Big|_0^1 + \frac{2}{5}x^{5/2}\Big|_0^1$

$= 3(1-0) + \frac{2}{5}(1^{5/2} - 0^{5/2}) = 3 + \frac{2}{5} = \frac{17}{5}$

Ex. $\int_0^2 (y-1)(2y+1)\,dy = \int_0^2 [2y^2 - y - 1]\,dy = \left[\frac{2}{3}y^3 - \frac{1}{2}y^2 - y\right]_0^2 = \frac{2}{3}(2^3 - 0^3) - \frac{1}{2}(2^2 - 0^2) - (2 - 0)$

$= \frac{2}{3}(8) - \frac{1}{2}(4) - 2 = \frac{16}{3} - 2 - 2 = \frac{16}{3} - \frac{12}{3} = \frac{4}{3}$

Ex. $\int_{-2}^{2} f(x)\,dx$, $f(x) = \begin{cases} 2 & -2 \le x \le 0 \\ 4 - x^2 & 0 < x \le 2 \end{cases}$



$\int_{-2}^{2} f(x)\,dx = \int_{-2}^{0} f(x)\,dx + \int_0^2 f(x)\,dx = \int_{-2}^{0} 2\,dx + \int_0^2 (4 - x^2)\,dx = 2x\Big|_{-2}^{0} + 4x\Big|_0^2 - \frac{1}{3}x^3\Big|_0^2$

$= 2(0 - -2) + 4(2-0) - \frac{1}{3}(2^3 - 0)$

$= 4 + 8 - \frac{8}{3} = \frac{36}{3} - \frac{8}{3} = \frac{28}{3}$

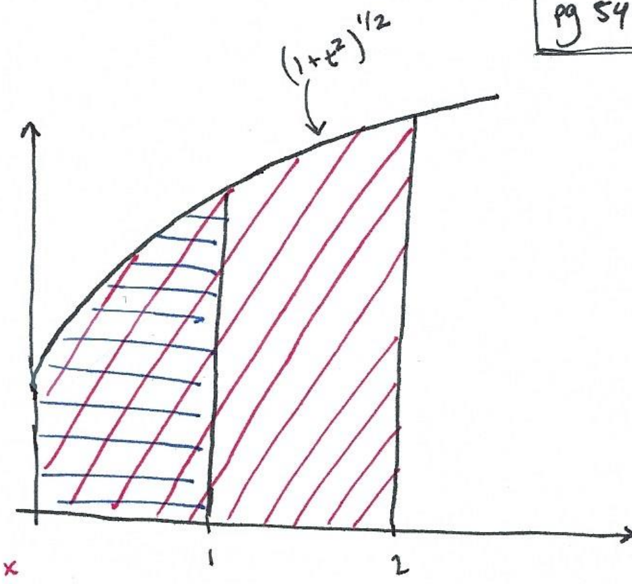
Fundamental Theorem of Calculus

Ex. Find the derivative of $g(x) = \int_0^x (1+t^2)^{1/2}\,dt$

$g(0) = \int_0^0 (1+t^2)^{1/2}\,dt = 0$

$g(1) = \int_0^1 (1+t^2)^{1/2}\,dt =$ blue area

$g(2) = \int_0^2 (1+t^2)^{1/2}\,dt =$ red area



By FTOC II, $g'(x) = (1+x^2)^{1/2}$

these can't be same letter.

t takes every value between 0 and x

Ex. Let $g(x) = \int_0^x \frac{1}{\sqrt{2+t^4}}\,dt$.

Then $g'(x) = \frac{1}{\sqrt{2+x^4}}$

Let $g_1(x) = \int_x^0 \frac{1}{\sqrt{2+t^4}}\,dt = -\int_0^x \frac{1}{\sqrt{2+t^4}}\,dt$

Then $g'(x) = \frac{-1}{\sqrt{2+x^4}}$

Define $h(x) = \int_0^{x^2} \frac{1}{\sqrt{2+t^4}}\,dt$

$g(x^2) = \int_0^{x^2} \frac{dt}{\sqrt{2+t^4}} = h(x)$

By chain rule,
$D[g(x^2)] = g'(x^2)(2x)$

$h'(x) = D[g(x^2)]$

$= g'(x^2)(2x) = \frac{2x}{\sqrt{2+(x^2)^4}} = \frac{2x}{\sqrt{2+x^8}}$

Define $k(x) = \int_0^{\tan x} (2+t^4)^{-1/2}\,dt = g(\tan x)$

$D[g(\tan x)] = g'(\tan x)\sec^2 x$

$k'(x) = g'(\tan x)\sec^2 x = (2+\tan^4 x)^{-1/2}\sec^2 x$