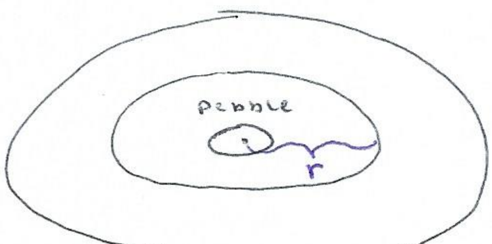
March 5, 2019 (89)

Example 3.7.2.

1) A pebble is dropped into a calm pond, causing ripples in the form of concentric circles. The radius r of the ripple is increasing at a constant rate of 1 foot per second. * When the radius is 4 feet, at what rate is the total area A of the disturbed water changing.



We know

* $\frac{dr}{dt} = 1$ ft/sec

we need

$\frac{dA}{dt}$ when $r = 4$ feet

FOR A CIRCLE

$A = \pi r^2 \rightarrow (A = \pi r^2)'$

derive, remember

$\frac{dA}{dt} = \pi \, 2r \frac{dr}{dt}$ — $(A)' = \frac{dA}{dt}$ since the derivative is in terms of t

π is a constant

and $(r^2)' = 2r\frac{dr}{dt}$

Since the derivative is in terms of t

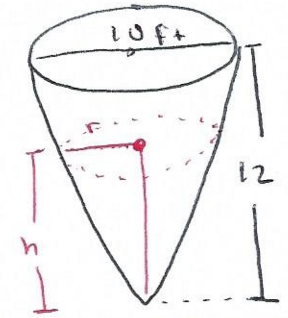
Substitute

$\frac{dA}{dt} = \pi \, 2 \underbrace{(4)}_{r = 4ft} \underbrace{(1)}_{\frac{dr}{dt} = 1\frac{ft}{sec}}$

$\boxed{\frac{dA}{dt} = 8\pi \, ft^2/sec}$ → Since

$\frac{dA}{dt} = \pi \cdot 2 \cdot 4 (ft) \cdot 1 \frac{(ft)}{(sec)} \rightarrow \frac{ft^2}{sec}$

2) A conical tank (with vertex down) is 10 feet across the top and 12 feet deep. Water is flowing into the tank at a rate of 10 cubic feet per minute. *
Find the rate of change of the depth of the water when the water is 8 feet deep.



We know

$\frac{dV}{dt} = 10 \frac{ft^3}{min}$ *

Find $\frac{dh}{dt}$ (h for height/depth)
when $h = 8$ ft.

VOLUME OF A CONE

$V = \frac{\pi}{3} r^2 h$ volume, radius, and height change with time. So V, r, h are all in terms of t.

Derive

$\left(V = \frac{\pi}{3} r^2 h\right)'$

$\frac{dV}{dt} = \frac{\pi}{3}\left(2r \frac{dr}{dt} \cdot h + r^2 \frac{dh}{dt}\right)$ → we need the chain rule to derive $r^2 h$ since both are in terms of t

we cannot substitute to find $\frac{dh}{dt}$ since we don't have r, $\frac{dr}{dt}$

Try to find a relation between r and h

Consider the ratio.

when , $h = 12$, $r = 5$ → $\frac{r}{h} = \frac{5}{12}$

then $\boxed{r = \frac{5}{12} h}$ → when both r and h are in terms of t.

So $\left(r = \frac{5}{12} h\right)' = \boxed{\frac{dr}{dt} = \frac{5}{12} \frac{dh}{dt}}$ Now we have values for r and $\frac{dr}{dt}$

$$\frac{dV}{dt} = \frac{\pi}{3}\left[2r\frac{dr}{dt}h + r^2\frac{dh}{dt}\right]$$

$$10 = \frac{\pi}{3}\left[2\cdot\left(\frac{5}{12}\right)h\cdot\frac{5}{12}\frac{dh}{dt}\cdot h + \left(\frac{5}{12}\right)^2 h^2\cdot\frac{dh}{dt}\right]$$

$$10 = \frac{\pi}{3}\left[\frac{50}{12}h^2\frac{dh}{dt} + \frac{25}{144}h^2\frac{dh}{dt}\right]$$

$$10\cdot\frac{3}{\pi} = \frac{50}{12}h^2\frac{dh}{dt} + \frac{25}{144}h^2\frac{dh}{dt}$$

$$10\cdot\frac{3}{\pi} = \frac{dh}{dt}\left[\frac{50}{12}h^2 + \frac{25}{144}h^2\right]$$

$$\frac{dh}{dt} = \frac{30}{\pi}\cdot\frac{1}{\frac{50}{12}h^2 + \frac{25}{144}h^2} = \boxed{\frac{30}{\pi}\cdot\frac{1}{\frac{50\cdot 64}{12} + \frac{25\cdot 64}{12}} = \frac{dh}{dt}}$$

when $h = 8$, $r = \frac{5}{12}h = \frac{40}{12}$, $\frac{dr}{dt} = \frac{5}{12}\frac{dh}{dt}$, $\frac{dV}{dt} = 10$

3) An airplane is flying on a flight path that will directly fly over a radar tracking station. The distance x is decreasing at a rate of 400 miles per hour when S = 10 miles. What is the speed of the plane? Find the rate of change in the angle of elevation of the radar.

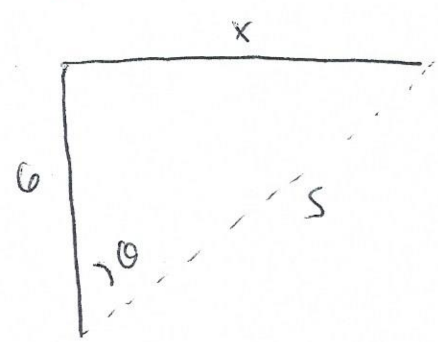
we know *

$$\frac{ds}{dt} = 400\ \frac{mi}{hr}$$

RATE of change in speed is $\frac{dv}{dt}$ since velocity is the derivative of position

we need

$\frac{dx}{dt}$ when $S = 10$



* picture provided *

a) what is the speed of the plane?

$s^2 - x^2 = 6^2$

$2s\frac{ds}{dt} - 2x\frac{dx}{dt} = 0$

$s = 10,\ \frac{ds}{dt} = -400$

$2(10)(-400) - 2x\frac{dx}{dt} = 0$

$s^2 - x^2 = 6^2$

$x = \sqrt{s^2 - 6^2} = \sqrt{10^2 - 6^2}$

$x = \sqrt{64} = 8$

$2(10)(-400) - 2(8)\frac{dx}{dt} = 0$

$-8000 = 16\frac{dx}{dt}$

$\frac{dx}{dt} = -\frac{16}{8000}$, $\boxed{\frac{dx}{dt} = -500 \text{ mi/hr}}$

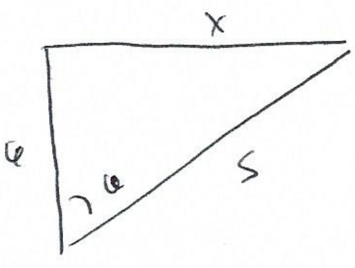
velocity

speed is |velocity|

so $\boxed{\text{Speed} = 500 \text{ mi/hr}}$

b)
Now what is the rate of change in the angle of the elevation of the radar?

refer to picture



→ we may write

$\sin\theta = \frac{x}{s}$ $\left(\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}\right)$

or

$\cos\theta = \frac{6}{s}$ $\left(\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}\right)$

or

$\tan\theta = \frac{x}{6}$ $\left(\tan\theta = \frac{\text{opposite}}{\text{adjacent}}\right)$

$\left(\tan\theta = \frac{x}{6}\right)' \to \sec^2\theta\frac{d\theta}{dt} = \frac{1}{6}\cdot\frac{dx}{dt}$ → since both are in terms of t

$\sec\theta = \frac{1}{\cos\theta}$

if $\cos\theta = \frac{6}{s}$,

then $\sec\theta = \frac{s}{6}$

$s = 10$

so $\sec\theta = \frac{10}{6} = \frac{5}{3}$

$\left(\frac{5}{3}\right)^2\frac{d\theta}{dt} = \frac{1}{6}\cdot(-500)$

$\frac{d\theta}{dt} = \frac{1}{6}\cdot(-500)\cdot\frac{9}{25}$

$\frac{d\theta}{dt} = \frac{-\cancel{500}^{20}\cdot\cancel{9}^{3}}{{}_{2}\cancel{6}\cdot\cancel{25}_{1}} = -\frac{60}{2} = \boxed{-30 = \frac{d\theta}{dt}}$