Example 3.7.2.

1) A pebble is dropped into a calm pond, causing ripples in the form of concentric circles. The radius $r$ of the ripple is increasing at a constant rate of 1 foot per second. When the radius is 4 feet, at what rate is the total area $A$ of the disturbed water changing?

We know:

* $\frac{dr}{dt} = 1 \text{ ft/sec}$

We need:

$\frac{dA}{dt}$ when $r = 4 \text{ feet}$

For a circle:

$A = \pi r^2 \Rightarrow A = \pi (r^2)$

Derive, remember:

$\frac{dA}{dt} = \pi 2r \frac{dr}{dt}$

Since the derivative is in terms of $t$:

$\pi$ is a constant

and $(r^2)' = 2r \frac{dr}{dt}$

Since the derivative is in terms of $t$:

Substitute:

$\frac{dA}{dt} = \pi 2(4)(1)$

$\frac{dr}{dt} = 1 \text{ ft/sec}$

$\frac{dA}{dt} = 8\pi (4^2)/\text{sec}$

→ Since

$\frac{dA}{dt} = 2 \cdot 4(1) \frac{64}{\text{sec}} \rightarrow \frac{ft^2}{\text{sec}}$
2) A conical tank (with vertex down) is 10 feet across the top and 12 feet deep. Water is flowing into the tank at a rate of 10 cubic feet per minute. Find the rate of change of the depth of the water when the water is 8 feet deep.

We know

\[ \frac{dV}{dt} = 10 \text{ ft}^3 \text{ min}^{-1} \]

Find \( \frac{dh}{dt} \) (change in height/depth)
when \( h = 8 \text{ ft} \).

**Volume of a Cone**

\[ V = \frac{\pi}{3} r^2 h \]

Derive

\[ V = \frac{\pi}{3} \left( 2r \frac{dr}{dt} h + r^2 \frac{dh}{dt} \right) \]

We cannot substitute to find \( \frac{dr}{dt} \) since we don't have \( \frac{r}{dt} \).

Try to find a relation between \( r \) and \( h \).

Consider the ratio when \( r = \frac{5}{12} h \) and \( h = 12 \), \( r = 5 \) and \( h = 12 \).

Then \( r = \frac{5}{12} h \) when both \( r \) and \( h \) are in terms of 6.

So \( (r = \frac{5}{12} h) \) implies \( \frac{dr}{dt} = \frac{\frac{5}{12} dh}{dt} \) and \( \frac{dh}{dt} = 12 \frac{dt}{dt} \). Now we have values for \( r \) and \( \frac{dr}{dt} \).
\[ \frac{dv}{dt} = \frac{n}{2} \left[ 2v \frac{dr}{dt} \cdot h + r^2 \frac{dh}{dt} \right] \]

\[ 10 = \frac{\pi}{3} \left[ 2 \left( \frac{5}{12} \right) h \cdot \frac{5}{12} \frac{dh}{dt} \cdot h + \left( \frac{5}{12} \right)^2 h^2 \cdot \frac{dh}{dt} \right] \]

\[ 10 = \frac{\pi}{3} \left[ \frac{50}{12} h^2 \frac{dh}{dt} + \frac{25}{144} h^2 \frac{dh}{dt} \right] \]

\[ 10 \cdot \frac{3}{\pi} = \frac{50}{12} h^2 \frac{dh}{dt} + \frac{25}{144} h^2 \frac{dh}{dt} \]

\[ 10 \cdot \frac{2}{\pi} = \frac{dh}{dt} \left[ \frac{50}{12} h^2 + \frac{25}{144} h^2 \right] \]

\[ \frac{dh}{dt} = \frac{30}{\pi} \cdot \frac{1}{\frac{50}{12} h^2 + \frac{25}{144} h^2} = \frac{30}{\pi} \cdot \frac{1}{\frac{50 \cdot 64}{12} + \frac{25 \cdot 64}{144}} = \frac{dr}{dt} \]

when \( h = 8 \), \( r = \frac{5}{12} h \), \( \frac{dr}{dt} = \frac{5}{12} \frac{dh}{dt} \), \( \frac{dv}{dt} = 10 \)

3) An airplane is flying on a flight path that will directly fly over a radar tracking station. The distance \( x \) is decreasing at a rate of 400 miles per hour when \( s = 10 \) miles. What is the speed of the plane? Find the rate of change in the angle of elevation of the radar.

We know
\[ \frac{ds}{dt} = 400 \text{ mi/hr} \]

We need
\[ \frac{dx}{dt} \text{ when } s = 10 \]

Rate of change in speed is \( \frac{dv}{dt} \), sine velocity is the derivative of position.

\[ \frac{dx}{dt} \]
a) What is the speed of the plane?

\[ s^2 - v^2 = u^2 \]

\[ 2s \frac{ds}{dt} - 2v \frac{dv}{dt} = 0 \]

\[ s = 10, \quad \frac{ds}{dt} = -400 \]

\[ 2(v(10))(-400) - 2x \frac{dx}{dt} = 0 \]

\[ s^2 - x^2 = u^2 \]

\[ x = \sqrt{s^2 - u^2} = \sqrt{10^2 - 8^2} \]

\[ x = \sqrt{36} = 6 \]

\[ 2(100)(-400) - 2(8) \frac{dx}{dt} = 0 \]

\[-8000 = 16 \frac{dx}{dt} \]

\[ \frac{dx}{dt} = -\frac{8000}{16} \]

\[ \frac{dx}{dt} = -500 \text{ mi/hr} \]

\[ \text{Velocity} \]

\[ \text{Speed} = 500 \text{ mi/hr} \]

b) Now what is the rate of change in the angle of the elevation of the radar?

Refer to picture.

\[ \sin \theta = \frac{x}{s} \] (\( \sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} \))

\[ \cos \theta = \frac{s}{s} \] (\( \cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} \))

\[ \tan \theta = \frac{\text{opposite}}{\text{adjacent}} \]

\[ \tan \theta = \frac{x}{s} \]

\[ \text{we may write} \]

\[ \sec \theta = \frac{1}{\cos \theta} \]

\[ \sec \theta = \frac{1}{\cos \theta} \]

\[ \text{is } \cos \theta = \frac{6}{5}, \]

\[ \text{then } \sec \theta = \frac{5}{6} \]

\[ s = 10 \]

\[ \text{so } \sec \theta = \frac{10}{6} \]

\[ \frac{\text{d} \theta}{\text{d} t} = \frac{1}{\frac{5}{6}} \cdot (-500) \]

\[ \frac{\text{d} \theta}{\text{d} t} = \frac{1}{\frac{5}{6}} \cdot (-500) \cdot \frac{9}{25} \]

\[ \frac{\text{d} \theta}{\text{d} t} = \frac{-20}{\frac{25}{9}} \cdot 3 \]

\[ \frac{\text{d} \theta}{\text{d} t} = \frac{-60}{2} = -30 \text{ degrees/hr} \]