Curve Sketching

Ex. $f(x) = \dfrac{x^2}{1-x^2}$

Domain: $(-\infty,-1) \cup (-1,1) \cup (1,\infty)$ $\longrightarrow$ vertical asymptotes @ $x = \pm 1$

$\lim\limits_{x\to 1^+} \dfrac{x^2}{1-x^2} = -\infty$ $\quad \dfrac{+1}{0}$ $\qquad \lim\limits_{x\to 1^-} \dfrac{x^2}{1-x^2} = \infty$

$\lim\limits_{x\to -1^+} \dfrac{x^2}{1-x^2} = \infty \qquad \lim\limits_{x\to -1^-} \dfrac{x^2}{1-x^2} = -\infty$

Vertical asymptotes

$\lim\limits_{x\to\infty} \dfrac{x^2}{1-x^2} = \lim\limits_{x\to\infty} \dfrac{1}{\frac{1}{x^2}-1} = -1$

$\lim\limits_{x\to-\infty} \dfrac{x^2}{1-x^2} = \lim\limits_{x\to-\infty} \dfrac{1}{\frac{1}{x^2}-1} = -1$

Horizontal asymptotes

$f'(x) = \left[x^2(1-x^2)^{-1}\right]' = 2x(1-x^2)^{-1} + (-1)x^2(1-x^2)^{-2}(-2x)$

$= 2x\left[\dfrac{1}{(1-x^2)} + \dfrac{x^2}{(1-x^2)^2}\right]$

$= 2x\left\{\dfrac{1-x^2+x^2}{(1-x^2)^2}\right\} = \dfrac{2x}{(1-x^2)^2}$

f' undefined at $x = \pm 1$

$f' = 0$ when $2x = 0$, $x = 0$

$f''(x) = \dfrac{2(1-x^2)^2 - (2x)(2)(1-x^2)(-2x)}{(1-x^2)^4}$

$= \dfrac{2(1-x^2)}{(1-x^2)^4}\left(1-x^2+4x^2\right) = \dfrac{2(1+3x^2)}{(1-x^2)^3}$

f'' not defined at $x = \pm 1$

$f'' \neq 0$ because $1 + 3x^2 > 0$

f'': A | + | A

$-2 \quad -1 \quad 0 \quad 1 \quad 2$

f': $\searrow \quad \searrow \quad \nearrow \quad \nearrow$

$-1 \quad 0 \quad 1$

f inc on $(0,1) \cup (1,\infty)$

f dec on $(-\infty,-1) (-1,0)$