Definite Integrals

Math 2413
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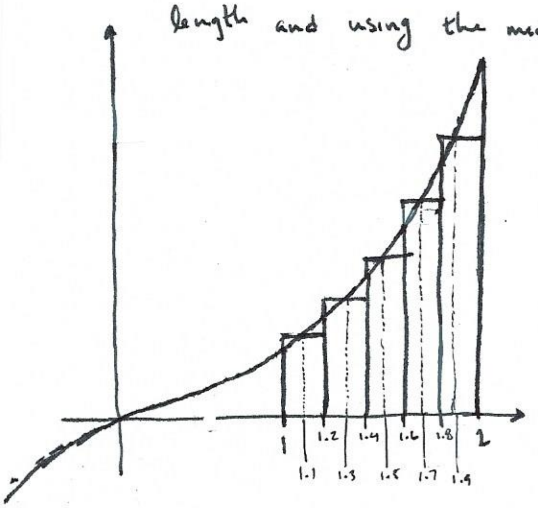
Ex. $f(x) = x^3$ on $[1, 2]$

Estimate the area under the curve (region bounded by $\begin{cases} y = x^3 \\ y = 0 \\ x = 1 \\ x = 2 \end{cases}$) by using 5 subintervals of equal length and using the midpoints of the intervals to obtain rectangle heights.



$n = 5$ rectangles (subintervals)

$\Delta x = \frac{b-a}{n} = \frac{2-1}{5} = \frac{1}{5}$

endpoints: $x_0 = 1$, $x_1 = 1.2$, $x_2 = 1.4$, $x_3 = 1.6$, $x_4 = 1.8$, $x_5 = 2$

midpoints:

$\left.\begin{array}{l} 1.1 = \bar{x}_1 \in [1, 1.2] = [x_0, x_1] \\ 1.3 = \bar{x}_2 \in [1.2, 1.4] = [x_1, x_2] \\ 1.5 = \bar{x}_3 \in [1.4, 1.6] = [x_2, x_3] \\ 1.7 = \bar{x}_4 \in [1.6, 1.8] = [x_3, x_4] \\ 1.9 = \bar{x}_5 \in [1.8, 2] = [x_4, x_5] \end{array}\right\}$ Sample points

Estimate: sum the area of the rectangles

* Σ means sum

$$\sum_{i=1}^{5} f(\bar{x}_i)\Delta x = f(\bar{x}_1)\Delta x + f(\bar{x}_2)\Delta x + f(\bar{x}_3)\Delta x + f(\bar{x}_4)\Delta x + f(\bar{x}_5)\Delta x$$

$$= (1.1)^3(.2) + (1.3)^3(.2) + (1.5)^3(.2) + (1.7)^3(.2) + (1.9)^3(.2)$$

This is a Riemann sum

$$= (.2)(1.331 + 2.197 + 3.375 + 4.913 + 6.859)$$

$$= (.2)(18.675)$$

$$= 3.735$$

Definite Integrals

Def. Suppose f is defined on the closed interval $[a,b]$.

Given a positive integer n, divide $[a,b]$ into n equal length subintervals.

Then $\Delta x = \frac{b-a}{n}$ is the length of each subinterval and the endpoints are

$$x_0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \quad \ldots, \quad x_n = a + n\Delta x = a + n\left(\frac{b-a}{n}\right) = a + b - a = b$$

Take a sample point $\overline{x}_i$ out of $[x_{i-1}, x_i]$ (the i^{th} subinterval).

Then $\sum_{i=1}^{n} f(\overline{x}_i)\Delta x$ is a Riemann Sum.

The <u>definite integral of f on $[a,b]$</u> is

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(\overline{x}_i)\Delta x = \int_a^b f(x)\,dx$$

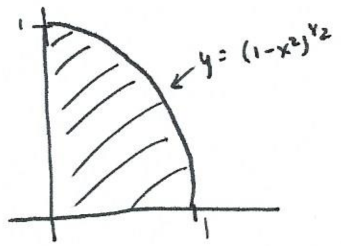
provide the limit exists. If the limit does exist, f is <u>integrable</u> on $[a,b]$.

Thm. If f is continuous on $[a,b]$, then f is integrable on $[a,b]$.

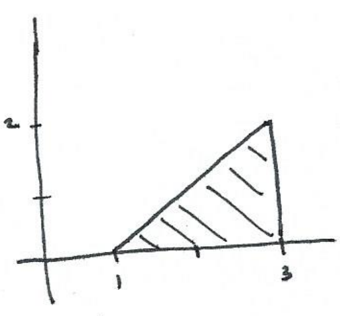
Thm. If f is continuous on $[a,b]$ and $f(x) \geq 0$ on $[a,b]$, then $\int_a^b f(x)\,dx$ = area bounded by the curves $\begin{cases} y = f(x) \\ y = 0 \\ x = a \\ x = b \end{cases}$

Ex. Compute $\int_0^1 (1-x^2)^{1/2}\,dx$ and $\int_0^3 (x-1)\,dx$ by interpreting these as areas of certain regions

$y = (1-x^2)^{1/2}$
$y^2 = 1 - x^2$
$x^2 + y^2 = 1$



$$\int_0^1 (1-x^2)^{1/2}\,dx = \frac{1}{4}\left[\pi(1)^2\right] = \frac{\pi}{4}$$



$$\int_0^3 (x-1)\,dx = \frac{1}{2}bh = \frac{1}{2}(2)(2) = 2$$