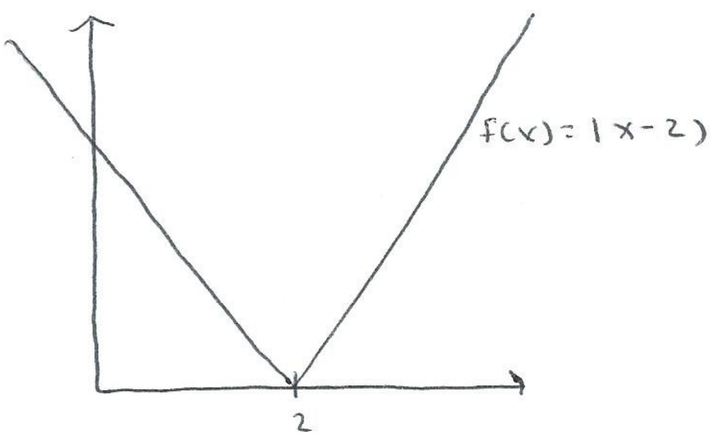
February 7, 2019

Example 3.1.9: Determine if the function is differentiable at the given point.

1) $f(x) = |x-2|$ at $x = 2$.



We can take the limit using

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

OR

$$f'(x) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

If $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$, then $\lim_{x \to 2} f(x)$ is the right/left limits.

Left-Hand

$$\lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2}$$

$|x-2| = -(x-2)$ for $x < 2$. sub for $x = 2$

$$= \lim_{x \to 2^-} \frac{-(x-2) - 0}{x-2}$$

$$= \lim_{x \to 2^-} \frac{-x+2}{x-2} = \lim_{x \to 2^-} \frac{-\cancel{(x-2)}}{\cancel{x-2}}$$

$$= \lim_{x \to 2^-} -1 = -1$$

→ this is actually the slope of the tangent line at this point, $x = 2$.

Right-Hand

$$\lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2}$$

$|x-2| = (x-2)$ for $x > 2$.

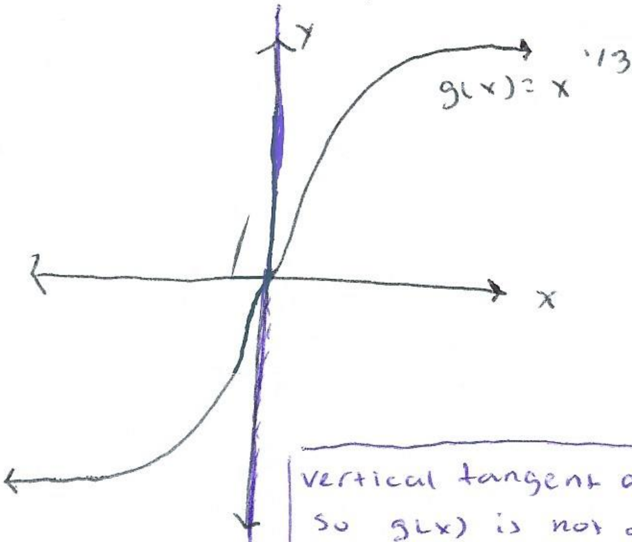
$$= \lim_{x \to 2^+} \frac{(x-2) - 0}{x-2}$$

$$= \lim_{x \to 2^+} 1 = 1$$

The left and right limits are NOT the same. This limit DNE for $x = 2$.

Example 3.1.9 cont.

2) $g(x) = x^{1/3}$ at $x = 0$.



By Def.
Take the limit

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0}$$

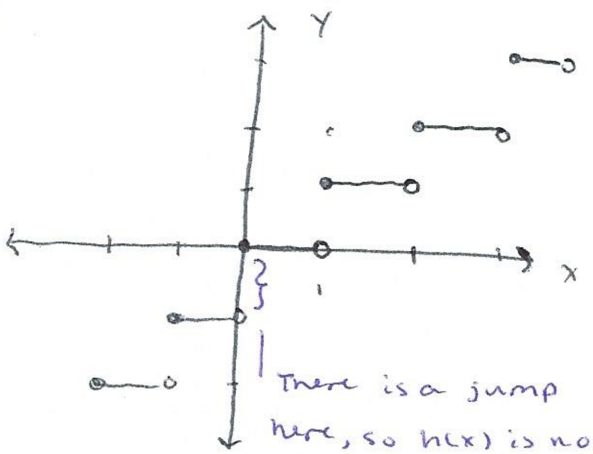
$$= \lim_{x\to 0} \frac{x^{1/3} - 0}{x}$$

$$= \lim_{x\to 0} \frac{1}{x^{2/3}} = \frac{1}{0} = \infty = \text{DNE}$$

The limit does not exist at $x = 0$. So the derivative doesn't exist at $x = 0$.

vertical tangent at $x = 0$, so $g(x)$ is not differentiable.

3) $h(x) = [\![x]\!]$ at $x = 0$.



There is a jump here, so $h(x)$ is not differentiable at $x = 0$.

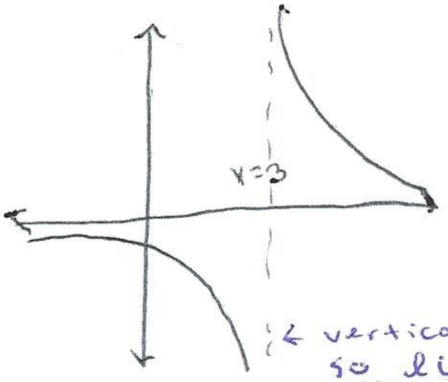
using def.

$$\lim_{x\to 0^-} h(x) = \lim_{x\to 0^-} \frac{1}{x} = -\infty \text{ DNE}$$

$$\lim_{x\to 0^+} h(x) = \lim_{x\to 0^+} \frac{1}{x} = \infty \text{ DNE}$$

Example 3.1.9

4) $\ell(x) = \dfrac{2}{x-3}$ at $x = 3$



$\ell(0) = \dfrac{2}{0} \Rightarrow$ DNE

← vertical asymptote so $\ell(x)$ is not differentiable at $x = 3$.

NOTE

- differentiable gives continuous
- not continuous gives not differentiable

5) $m(x) = \begin{cases} x^2 + 1, & x \leq 2 \\ 4x - 3, & x > 2 \end{cases}$ at $x = 2$.

Left-hand

$\displaystyle\lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2}$ sub for $f(x) = x^2 + 1$, $f(2) = 5$

$\displaystyle = \lim_{x \to 2^-} \frac{x^2 + 1 - 5}{x - 2}$ simplify

$\displaystyle = \lim_{x \to 2^-} \frac{x^2 - 4}{x - 2}$ factor

$\displaystyle = \lim_{x \to 2^-} \frac{(x+2)\cancel{(x-2)}}{\cancel{(x-2)}}$ cancel + direct sub

$\displaystyle = \lim_{x \to 2^-} x + 2 = 4$

Right-hand

$\displaystyle\lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2}$ sub for $f(x) = 4x - 3$, $f(2) = 5$

$\displaystyle = \lim_{x \to 2^+} \frac{4x - 3 - 5}{x - 2}$ simplify

$\displaystyle = \lim_{x \to 2^+} \frac{4x - 8}{x - 2}$ factor

$\displaystyle = \lim_{x \to 2^+} \frac{4\cancel{(x-2)}}{\cancel{(x-2)}}$ cancel + direct sub

$\displaystyle = \lim_{x \to 2^+} 4 = 4$

$$4 = \lim_{x \to 2^-} m(x) = \lim_{x \to 2^+} m(x).$$

Therefore $\displaystyle\lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = 4$.

Notice, $\displaystyle f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$.

So f is differentiable at $x = 2$.

3.2 Basic Differentiation Rules and Rates of Change

THEOREM 3.2.1

1. Let c and n be real numbers. If f and g are differentiable functions, then

   (a) $(c)' = 0$

   (b) $(x^n)' = nx^{n-1}$

   (c) $(cf(x))' = cf'(x)$

   (d) $(f(x) \pm g(x))' = f'(x) \pm g'(x)$

2. $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$, $(e^x)' = e^x$

Example 3.2.2 Find the derivative of the function.

(a) $g(x) = \sqrt[4]{x^5} - \frac{6}{x^3} + x^{-2} + 4$ → a sum, so we differentiate each part to get the derivative via Theorem 3.2.1 (d)

$g'(x) = (\sqrt[4]{x^5})' - \left(\frac{6}{x^3}\right)' + (x^{-2})' + (4)'$

$g'(x) = (x^{\frac{5}{4}})' - (6(x^{-3}))' + (x^{-2})' + (4)'$ → Rewrite

$g'(x) = \underbrace{\frac{5}{4}x^{\frac{1}{4}}}_{\text{use (b)}} - \underbrace{6(-3x^{-4})}_{\text{use (c),(b)}} - \underbrace{2x^{-3}}_{\text{use (b)}} + \underbrace{0}_{\text{use (a)}}$ use the rules from Theorem 3.2.1 to derive

$\boxed{g'(x) = \frac{5}{4}x^{\frac{1}{4}} + 18x^{-4} - 2x^{-3}}$

(b) $h(x) = \frac{3}{2\sqrt[3]{x^2}} - \frac{\sin x}{5} + 3e^x - \pi^2$

$h'(x) = \left(\frac{3}{2\sqrt[3]{x^2}}\right)' - \left(\frac{\sin x}{5}\right)' + (3e^x)' - (\pi^2)'$

$h'(x) = \frac{3}{2}\left(\frac{1}{\sqrt[3]{x^2}}\right)' - \frac{1}{5}(\sin x)' + 3(e^x)' - (\pi^2)'$

$h'(x) = \frac{3}{2}\left(\frac{1}{x^{2/3}}\right)' - \frac{1}{5}(\sin x)' + 3(e^x)' - (\pi^2)'$

$h'(x) = \frac{3}{2}(x^{-2/3})' - \frac{1}{5}(\sin x)' + 3(e^x)' - (\pi^2)'$

$h'(x) = \frac{3}{2}\left(-\frac{2}{3}x^{-\frac{2}{3}-\frac{3}{3}}\right) - \frac{1}{5}(\cos x) + 3(e^x) - 0$

$\boxed{h'(x) = -x^{-5/3} - \frac{\cos x}{5} + 3e^x}$