Ex. $f(x) = \frac{x^2}{x^2 + 3}$

Find local max/min; intervals where $f$ inc/dec; domain; asymptotes.

Give rough sketch of graph

Domain: $\mathbb{R}$ = all reals

Asymptotes: no vertical

$\lim_{x \to 0} \frac{1}{x^2 + 3} = 0$

$\lim_{x \to \pm \infty} \frac{x^2}{x^2 + 3} = 1$

$x$-intercept: 0

$y$-intercept: 0

$f'(x) = \frac{2x(x^2 + 3) - x^2 (2x)}{(x^2 + 3)^2}$

$f'(x) = \frac{6x}{(x^2 + 3)^2}$

$f'$ exists on $\mathbb{R}$

$x = 0$ is the only crit. pt.

$f''(x) = 0 \Rightarrow 6x = 0 \Rightarrow x = 0$

$f$ dec on $(-\infty, 0)$

$f$ inc on $(0, \infty)$

Def. If the graph of $f$ lies above all of its tangent lines on an interval $I$, then $f$ is concave up on $I$.

Def. If the graph of $f$ lies below all of its tangent lines on an interval $I$, then $f$ is concave down on $I$.

Note: 1st derivative $f'$ gives information about increasing, decreasing, local max, local min.

2nd derivative $f''$ gives information about concavity and inflection points.
Def. A point $p$ on a curve $f(x)$ is an inflection point if $f$ is continuous there and the curve changes concavity at $p$.

**Concavity test:**
1. $f'' > 0$ on $I$, then $f$ concave up on $I$
2. $f'' < 0$ on $I$, then $f$ concave down on $I$

**Example**

$f'(x) = \frac{6x}{(x^2+3)^2} \Rightarrow f''(x) = \frac{6(x^2+3)^3 - 6x(x)(x^2+3)(2x)}{(x^2+3)^4} = \frac{6(x^2+3)^3 - 24x^2}{(x^2+3)^4} = \frac{6(x^2+3) - 24x^2}{(x^2+3)^3} = \frac{6x^2 + 18 - 24x^2}{(x^2+3)^3}$

$f'' = \frac{18(1-x^2)}{(x^2+3)^3}$

$f''(0) = 0 \Rightarrow 1 - x^2 = 0 \Rightarrow x = \pm 1$

**Exercise**

Discuss $y = x^4 - 4x^3$ wrt:
- concavity, inflection points, local max/min, intervals where $f$ inc/dec.

Sketch the graph of $y$:

- $y' = 4x^3 - 12x^2 = 4x^2(x - 3)$
- $y'' = 12x^2 - 24x = 12x(x - 2)$
- $y'' = \text{defined everywhere}$
- $y'' = 0 \Rightarrow x = 0, 2$

$y$ inc on $(0, 3)$

$y$ concave up on $(-\infty, 0) \cup (2, \infty)$

$y$ concave down on $(0, 2)$

$y$ dec on $(-\infty, 2) \cup (3, \infty)$

$y$ inflection points:

$(1, y(1)) = (2, -16)$

$(0, y(0)) = (0, 0)$

$y$ local max:

$y(3) = 3^4 - 4(3)^3 = 27$