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2. (i) Find the explicit function by solving the equation for y in terms of x.

(ii) Find $\frac{dy}{dx}$ implicitly

(iii) Show that the results are the same.

if $x^2 + y^2 = 1$

(i) $y^2 = 1 - x^2$

$$y = \sqrt{1-x^2} = (1-x^2)^{1/2}$$

→ derive using the Chain Rule

$$y' = \tfrac{1}{2}(1-x^2)^{-1/2}(-2x)$$

$$y' = \tfrac{1}{2}(-2x)(1-x^2)^{-1/2}$$

$$y' = -x(1-x^2)^{-1/2}$$

$$y' = -\frac{x}{\sqrt{1-x^2}} \xrightarrow[y=\sqrt{1-x^2}]{} -\frac{x}{y}$$

(ii) $D[x^2 + y^2 = 1]$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

$D(y^2) = 2y\frac{dy}{dx}$ since we derived y^2 in terms of x.

(iii) It has been shown both methods yield the same answer.

Find $\frac{dy}{dx}$ when $y^3-5y-x^2=-4+\ln(xy)$

$\ln(xy)=\ln(x)+\ln(y)$ so

$y^3-5y-x^2=-4+\ln(x)+\ln(y)$ $\quad D(\ln x)=\frac{1}{x}$

Derive $\quad D(\ln y)=\frac{1}{y}\frac{dy}{dx}$

$3y^2\frac{dy}{dx}-5\frac{dy}{dx}-2x=0+\frac{1}{x}+\frac{1}{y}\frac{dy}{dx}$

$3y^2\frac{dy}{dx}-5\frac{dy}{dx}-\frac{1}{y}\frac{dy}{dx}=\frac{1}{x}+2x \longleftarrow$ collect all $\frac{dy}{dx}$ to the left side and solve

$\frac{dy}{dx}\left(3y^2-5-\frac{1}{y}\right)=\frac{1}{x}+2x \longleftarrow$ factor out $\frac{dy}{dx}$ and divide to get $\frac{dy}{dx}$

$$\frac{dy}{dx}=\frac{\frac{1}{x}+2x}{3y^2-5-\frac{1}{y}}$$

Find $\frac{dy}{dx}$ when $\underset{\substack{\text{chain rule}\\ \text{product rule}}}{\sqrt{xy}}=\overset{\text{product rule}}{x^2y}-\underset{\text{product rule}}{e^{xy}}$

$(\sqrt{xy}=x^2y-e^{xy})'=(\sqrt{xy})'=(x^2y)'-(e^{xy})'$

$(\sqrt{xy})'=((xy)^{1/2})'$ → rewrite and use Chain Rule

$=\frac{1}{2}(xy)^{-1/2}(xy)'$ → $(xy)'$ requires product rule

$=\frac{1}{2}(xy)^{-1/2}(x'y+xy')=\frac{1}{2}(xy)^{-1/2}\left(y+x\frac{dy}{dx}\right)$

$(\sqrt{xy})'=\frac{y+x\frac{dy}{dx}}{2\sqrt{xy}}=\frac{y}{2\sqrt{xy}}+\frac{x\frac{dy}{dx}}{2\sqrt{xy}}$ → separate the fraction to get $\frac{dy}{dx}$ on one term

$(x^2y)'=(x^2)'y+x^2y'$ → use product rule

$(x^2y)'=2xy+x^2\frac{dy}{dx}$ $\longrightarrow$

cont.

Find $\frac{dy}{dx}$ when $\sqrt{xy} = x^2y - e^{xy}$

we found

$$(\sqrt{xy})' = \frac{y}{2\sqrt{xy}} + \frac{x}{2\sqrt{xy}}\frac{dy}{dx}$$

$$(x^2y)' = 2xy + x^2\frac{dy}{dx}$$

now

$$(e^{xy})' = (xy)'(e^{xy})$$

$$= (x'y + xy')(e^{xy})$$

$$= \left(y + x\frac{dy}{dx}\right)(e^{xy})$$

$$(e^{xy})' = ye^{xy} + e^{xy}x\frac{dy}{dx}$$

so $(\sqrt{xy})' = (x^2y)' - (e^{xy})'$

is

$$\frac{y}{2\sqrt{xy}} + \frac{x}{2\sqrt{xy}}\frac{dy}{dx} = 2xy + x^2\frac{dy}{dx} - \left(ye^{xy} + e^{xy}x\frac{dy}{dx}\right)$$

$$\frac{y}{2\sqrt{xy}} + \frac{x}{2\sqrt{xy}}\frac{dy}{dx} = 2xy + x^2\frac{dy}{dx} - ye^{xy} - x\frac{dy}{dx}e^{xy}$$

→ collect all $\frac{dy}{dx}$

$$\frac{x}{2\sqrt{xy}}\frac{dy}{dx} - x^2\frac{dy}{dx} + x\frac{dy}{dx}e^{xy} = 2xy - \frac{y}{2\sqrt{xy}} - ye^{xy}$$

→ factor out $\frac{dy}{dx}$ and solve

$$\frac{dy}{dx}\left(\frac{x}{2\sqrt{xy}} - x^2 + xe^{xy}\right) = 2xy - \frac{y}{2\sqrt{xy}} - ye^{xy}$$

$$\frac{dy}{dx} = \frac{2xy - \frac{y}{2\sqrt{xy}} - ye^{xy}}{\frac{x}{2\sqrt{xy}} - x^2 + xe^{xy}}$$

Find $\frac{dy}{dx}$ when $\tan(x+y) = \sin x + 2\cos y$

Recall

$(\tan a)' = \sec^2 a$

$(\sin a)' = \cos a$

$(\cos a)' = -\sin a$

$$(\tan(x+y))' = (\sin x)' + (2\cos y)'$$

Chain Rule

$$\underbrace{\sec^2(x+y) \cdot (x+y)'} = \cos x + (2\cos y)'$$

$$\sec^2(x+y)(x'+y') = \cos x + (2\cos y)'$$

$$\sec^2(x+y)\left(1+\frac{dy}{dx}\right) = \cos x + (2\cos y)' \rightarrow$$ 2 x derivative of cos y, $-\sin y \frac{dy}{dx}$, since this is y in terms of x

$$\sec^2(x+y)\left(1+\frac{dy}{dx}\right) = \cos x + 2\cdot\left(-\sin y \cdot \frac{dy}{dx}\right)$$

$$\sec^2(x+y) + \sec^2(x+y)\frac{dy}{dx} = \cos x - 2 \cdot \sin y \cdot \frac{dy}{dx}$$ simplify, distribut

$$\sec^2(x+y)\frac{dy}{dx} + 2\sin y \frac{dy}{dx} = \cos x - \sec^2(x+y)$$ collect all $\frac{dy}{dx}$ and solve

$$\frac{dy}{dx}\left(\sec^2(x+y) + 2\sin y\right) = \cos x - \sec^2(x+y)$$

$$\frac{dy}{dx} = \frac{\cos x - \sec^2(x+y)}{\sec^2(x+y) + 2\sin y}$$

Consider $y = x^{2x}$. Can you derive it?
we have the rules $(x^n)' = nx^{n-1}$ and $(a^x)' = a^x \ln a$,
however, x^{2x} satisfies neither of these terms.

Try natural logarithm.

$$y = x^{2x} \rightarrow \ln(y) = \ln(x^{2x}) \rightarrow \ln(x^{2x}) = 2x\ln(x)$$

$$\ln(y) = 2x\ln(x)$$ Now try to derive.

$$(\ln(y))' = (2x\ln(x))' \rightarrow$$ product rule

remember

$(\ln x)' = \frac{1}{x}$

we can rewrite y in terms of x

$$\frac{1}{y}\cdot\frac{dy}{dx} = (2x)'(\ln(x)) + (2x)(\ln(x))'$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = 2\ln(x) + 2x \cdot \frac{1}{x}$$

$$\frac{dy}{dx} = \boxed{y}(2\ln(x) + 2) = \boxed{x}^{2x}(2\ln(x)+2)$$

Find the derivative of $y=(\ln x)^{\ln x}$.

$y=(\ln x)^{\ln x}$ → apply natural logarithm

$\ln(y)=\ln(\ln x^{\ln x})$

$(\ln(y))'=(\ln x\cdot\ln(\ln x))'$ → use product rule

$\frac{1}{y}\cdot\frac{dy}{dx}=(\ln x)'(\ln(\ln x))+(\ln x)(\ln(\ln x))'$

$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{1}{x}\cdot\ln(\ln x)+\ln x\cdot\frac{1}{\ln x}\cdot\frac{1}{x}$ since $(\ln x)'=\frac{1}{x}$, $(\ln(\ln x))'=\frac{1}{\ln x}$

$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{1}{x}\cdot\ln(\ln x)+\frac{1}{x}$ → simplify and solve for $\frac{dy}{dx}$

$\frac{dy}{dx}=y\left(\frac{1}{x}\cdot\ln(\ln x)+\frac{1}{x}\right)$

This answer is fine, but we can also rewrite y so that $\frac{dy}{dx}$ is in terms of x.

Since $y=(\ln x)^{\ln x}$, $\frac{dy}{dx}=(\ln x)^{\ln x}\left(\frac{1}{x}\cdot\ln(\ln x)+\frac{1}{x}\right)$

Now try to use e rather than natural logarithm

$y=(\ln x)^{\ln x}\rightarrow e^{y}=e^{\ln x^{\ln x}}$

$(e^{y})'=((e^{\ln x})^{\ln x})'$

$e^{y}\cdot\frac{dy}{dx}=e^{\ln x^{\ln x}}\cdot\underbrace{[(\ln x)^{1/x}]'}$

we don't know how to derive this. So this method may not be the best for this type of problem.