2. (i) Find the explicit function by solving the equation for \( y \) in terms of \( x \).

(ii) Find \( \frac{dy}{dx} \) implicitly.

(iii) Show that the results are the same.

If \( x^2 + y^2 = 1 \)

\[(i) \quad y^2 = 1 - x^2 \]

\[y = \sqrt{1-x^2} = (1-x^2)^{1/2} \]

\[y' = \frac{1}{2} (1-x^2)(-2x) \]

\[x' = \frac{1}{2} (-2x)(1-x^2)^{-1/2} \]

\[y' = -x (1-x^2)^{-1/2} \]

\[y' = -x \sqrt{\frac{1}{1-x^2}} \quad \frac{y}{y} \]

\[(ii) \quad D[x^2 + y^2 = 1] \]

\[2x + 2y \frac{dy}{dx} = 0 \]

\[2y \frac{dy}{dx} = -2x \]

\[\frac{dy}{dx} = -\frac{x}{y} \]

\[(iii) \quad \text{It has been shown both methods yield the same answer.} \]
Find \( \frac{dy}{dx} \) when \( y^3 - 5y - x^2 = -4 + \ln(xy) \)

\[
\ln(xy) = \ln(x) + \ln(y) \quad \text{so}
\]

\[
y^3 - 5y - x^2 = -4 + \ln(x) + \ln(y)
\]

\[
\text{Derive}
\]

\[
3y^2 \frac{dy}{dx} - 5 \frac{dy}{dx} - 2x = 0 + \frac{1}{x} + \frac{1}{y} \frac{dy}{dx}
\]

\[
3y^2 \frac{dy}{dx} - 5 \frac{dy}{dx} - \frac{1}{y} \frac{dy}{dx} = \frac{1}{x} + 2x
\]

\[
\frac{dy}{dx} (3y^2 - 5 - \frac{1}{y}) = \frac{1}{x} + 2x
\]

\[
\frac{dy}{dx} = \frac{\frac{1}{x} + 2x}{3y^2 - 5 - \frac{1}{y}}
\]

---

Find \( \frac{dy}{dx} \) when \( \sqrt[3]{xy} = \frac{x^2y - e^{xy}}{1} \)

\[
\text{product rule}
\]

\[
\left( \sqrt[3]{xy} \right)' = \left( \sqrt[3]{xy} \right)' = \left( x^2y \right)' - (e^{xy})'
\]

\[
\left( \sqrt[3]{xy} \right)' = \left( (xy)^{\frac{1}{3}} \right)' \quad \text{rewrites and use chain rule}
\]

\[
= \frac{1}{2} (xy)^{-\frac{2}{3}} (xy)' + (xy)' \quad \text{requires product rule}
\]

\[
= \frac{1}{2} (xy)^{-\frac{1}{3}} (x'y + xy') = \frac{1}{2} (xy) \left( y + x \frac{dy}{dx} \right)
\]

\[
\left( \sqrt[3]{xy} \right)' = y + x \frac{dy}{dx}
\]

\[
\frac{\frac{dy}{dx}}{2Nxy} = \frac{y}{2Nxy} + \frac{x \frac{dy}{dx}}{2Nxy}
\]

\[
\text{separate the equation to get} \quad \frac{dy}{dx} \quad \text{on one term}
\]

\[
(x^2y)' = (x^2)'y + x^2y' \quad \text{use product rule}
\]

\[
(x^2y)' = 2xy + x^2 \frac{dy}{dx}
\]

cont.
Find \( \frac{dy}{dx} \) when \( \sqrt{xy} = x^2y - e^{xy} \)

We found
\[
\left( \sqrt{xy} \right)' = \frac{y}{2\sqrt{xy}} + \frac{x}{2\sqrt{xy}} \frac{dy}{dx}
\]
\[
(x^2y)' = 2xy + x^2 \frac{dy}{dx}
\]

Now
\[
(e^{xy})' = (xy)'(e^{xy})
\]
\[
= (x'y + xy')e^{xy}
\]
\[
= (y + x \frac{dy}{dx})e^{xy}
\]
\[
(e^{xy})' = ye^{xy} + e^x \frac{dy}{dx}
\]

So
\[
\left( \sqrt{xy} \right)' = (x^2y)' - (e^{xy})'
\]

is
\[
\frac{y}{2\sqrt{xy}} + \frac{x}{2\sqrt{xy}} \frac{dy}{dx} = 2xy + x^2 \frac{dy}{dx} - (ye^{xy} + e^x \frac{dy}{dx})
\]
\[
\frac{y}{2\sqrt{xy}} + \frac{x}{2\sqrt{xy}} \frac{dy}{dx} = 2xy + x^2 \frac{dy}{dx} - ye^{xy} - e^x \frac{dy}{dx}
\]
\[
\frac{x}{2\sqrt{xy}} \frac{dy}{dx} = x^2 \frac{dy}{dx} + x \frac{dy}{dx} - ye^{xy} - e^x \frac{dy}{dx}
\]
\[
\frac{dy}{dx} \left( \frac{x}{2\sqrt{xy}} - x^2 + xe^{xy} \right) = 2xy - \frac{y}{2\sqrt{xy}} - ye^{xy}
\]
\[
\frac{dy}{dx} = \frac{2xy - \frac{y}{2\sqrt{xy}} - ye^{xy}}{\frac{x}{2\sqrt{xy}} - x^2 + xe^{xy}}
\]
Find \( \frac{dy}{dx} \) when \( \tan(x+y) = \sin x + 2 \cos y \)

\[
\begin{align*}
(tan(x+y))' &= (\sin x)' + (2 \cos y)' \\
&= \cos x + (2 \cos y)' \\
&= \cos x + (2 \cos y)'
\end{align*}
\]

\[
\begin{align*}
\text{Chain Rule} \\
\sec^2(x+y) \cdot (x+y)' &= \cos x + (2 \cos y)' \\
\sec^2(x+y)(1 + \frac{dy}{dx}) &= \cos x + (2 \cos y)' \\
\sec^2(x+y)(1 + \frac{dy}{dx}) &= \cos x + 2 \cdot (-\sin y \cdot \frac{dy}{dx}) \\
\text{Simplify, distrib} \\
\sec^2(x+y) \frac{dy}{dx} + 2 \sin y \frac{dy}{dx} &= \cos x - \sec^2(x+y) \text{ collect all } \frac{dy}{dx} \\
\frac{dy}{dx} (\sec^2(x+y) + 2 \sin y) &= \cos x - \sec^2(x+y) \\
\frac{dy}{dx} &= \frac{\cos x - \sec^2(x+y)}{\sec^2(x+y) + 2 \sin y}
\end{align*}
\]

Consider \( y = x^{2x} \). Can you derive it?

we have the rules \((x^n)' = nx^{n-1}\) and \((a^x)' = a^x \ln a\), however, \(x^{2x}\) satisfies neither of these terms.

Try natural logarithm.

\[
\begin{align*}
y &= x^{2x} \\
\ln(y) &= \ln(x^{2x}) = 2x \ln(x)
\end{align*}
\]

\[
\begin{align*}
\ln(y) &= 2x \ln(x) \text{ now try to derive} \\
(\ln(y))' &= (2x \ln(x))' \text{ product rule} \\
\frac{1}{y} \cdot \frac{dy}{dx} &= (2x)' (\ln(x)) + (2x) (\ln(x))' \\
\frac{1}{y} \cdot \frac{dy}{dx} &= 2 \ln(x) + 2x \cdot \frac{1}{x} \\
\frac{dy}{dx} &= y \cdot (2 \ln(x) + 2) = x^{2x} \cdot (2 \ln(x) + 2)
\end{align*}
\]
Find the derivative of \( y = (\ln x)^{\ln x} \).

\[ y = (\ln x)^{\ln x} \rightarrow \text{apply natural logarithm} \]

\[ \ln(y) = \ln((\ln x)^{\ln x}) \]

\[ (\ln(y))' = (\ln x)'(\ln(\ln x)) + (\ln x)(\ln(\ln x))' \text{ → use product rule} \]

\[ \frac{1}{y} \cdot \frac{dy}{dx} = (\ln x)'(\ln(\ln x)) + (\ln x)(\ln(\ln x))' \]

\[ \frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{x} \cdot \ln(\ln x) + \ln x \cdot \frac{1}{\ln x} \]

\[ \frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{x} \cdot \ln(\ln x) + \frac{1}{x} \]

\[ \frac{dy}{dx} = y \left( \frac{1}{x} \cdot \ln(\ln x) + \frac{1}{x} \right) \]

This answer is fine, but we can also rewrite \( y \) so that \( \frac{dy}{dx} \) is in terms of \( x \).

Since \( y = (\ln x)^{\ln x} \), \[ \frac{dy}{dx} = (\ln x)^{\ln x} \left( \frac{1}{x} \cdot \ln(\ln x) + \frac{1}{x} \right) \]

Now try to use \( e \) rather than natural logarithm.

\[ y = (\ln x)^{\ln x} \rightarrow e^y = \ln x \]

\[ (e^y)' = ((\ln x)^{\ln x})' \]

\[ e^y \cdot \frac{dy}{dx} = e^{\ln x} \cdot [\ln x]^y \cdot [\ln x] \]

We don't know how to derive this. So this method may not be the best for this type of problem.