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Def 3.4.4

1. If a is a positive real number ($a \neq 1$) and x is any real number, then the exponential function to the base, a, is denoted by

$$\boxed{a^x}$$

and is defined by

$$\boxed{a^x = e^{(\ln a)x}}.$$

If $a = 1$, then $y = a^x = 1^x = 1$, which is a constant function.

2. If a is a positive real number ($a \neq 1$) and x is any real positive number, then the logarithmic function to the base, a, is denoted by

$$\boxed{\log_a x}$$

and is defined as

$$\boxed{\log_a x = \frac{1}{\ln a} \ln x.}$$

Theorem 3.4.5

Let a be a positive real number ($a \neq 1$) and let $f(x)$ be a differentiable function of x. Then

$$(a^x)' = (\ln a)a^x \qquad (a^{f(x)})' = (\ln a)a^{f(x)} f'(x)$$

$$(\log_a x)' = \frac{1}{(\ln a)x} \qquad (\log_a f(x))' = \frac{1}{(\ln a)f(x)} f'(x)$$

Example 3.4.4

1. Find the derivative of m if $m(x) = \log_{10} \cos\left(\frac{x}{2}\right) + 2^{3x}$

3.4.5 says $\left(\log_a f(x)\right)' = \frac{1}{\ln a \cdot f(x)} \cdot f'(x)$

and $(a^x)' = (\ln a) a^x$

$m'(x) = \underbrace{\left(\log_{10} \cos\left(\frac{x}{2}\right)\right)'}_{} + \underbrace{\left(2^{3x}\right)'}_{}$

$m'(x) = \underbrace{\frac{1}{\ln 10 \cdot \cos \frac{x}{2}} \left(\cos \frac{x}{2}\right)'}_{} + \underbrace{\ln 2 \cdot 2^{3x} \cdot (3x)'}_{}$

$m'(x) = \underbrace{\frac{1}{\ln 10 \cdot \cos \frac{x}{2}} \left(-\sin \frac{x}{2}\right)\left(\frac{x}{2}\right)'}_{} + \underbrace{\ln 2 \cdot 2^{3x} \cdot 3}_{}$

$m'(x) = \underbrace{\frac{1}{\ln 10 \cdot \cos \frac{x}{2}} \left(-\sin \frac{x}{2}\right)\left(\frac{1}{2}\right)}_{} + \underbrace{\ln 2 \cdot 2^{3x} \cdot 3}_{}$

2. Given that $g(5) = -3$, $g'(5) = 6$, $h(5) = 3$, $h'(5) = -2$, find $f'(5)$, if possible.

(a) $f(x) = g(x)h(x)$ → use product rule

$f'(x) = g'(x)h(x) + g(x)h'(x)$

find $f'(5)$

$f'(5) = g'(5)h(5) + g(5)h'(5)$

$f'(5) = (6 \cdot 3) + (-3 \cdot -2)$

$f'(5) = 18 + 6 = 24.$

(b) $f(x) = g^3(x)$. rewrite to suit the chain Rule

$f(x) = [g(x)]^3$

$f'(x) = [g(x)]^{3\,'} = \underset{\substack{\uparrow \\ \text{expo } g(x)}}{3} \underset{\substack{\uparrow \\ g(x)^{n-1}}}{[g(x)]^2} \cdot \underset{\substack{\uparrow \\ g'(x)}}{g'(x)}$

$f'(x) = 3[g^2(x)] \cdot g'(x)$

find $f'(5)$

$f'(5) = 3[g(5)]^2 \cdot g'(5)$

$f'(5) = 3 \cdot (-3)^2 \cdot 6 = 162.$

(c) $f(x) = \ln\left(\frac{3g(x)}{h(x)}\right)^2$

If we dont remember how to derive for ln,

$$f'(x) = \left[\ln\left(\frac{3g(x)}{h(x)}\right)^2\right]'$$

→ use chain rule
$D[\ln f^2(x)] = \frac{1}{f^2(x)} \cdot [f^2(x)]'$

$$f'(x) = \frac{1}{\left(\frac{3g(x)}{h(x)}\right)^2} \cdot \left[\left(\frac{3g(x)}{h(x)}\right)^2\right]'$$

→ use chain rule
$D[f^2(x)] = 2 \cdot f(x) \cdot f'(x)$

$$f'(x) = \frac{1}{\left(\frac{3g(x)}{h(x)}\right)^2} \cdot \underbrace{2\left(\frac{3g(x)}{h(x)}\right) \cdot \left(\frac{3g(x)}{h(x)}\right)'}_{\text{quotient rule}}$$

$$f'(x) = \frac{1}{\left(\frac{3g(x)}{h(x)}\right)^2} \cdot 2\left(\frac{3g(x)}{h(x)}\right) \cdot \left(\frac{3g'(x) \cdot h(x) - 3g(x)h'(x)}{h^2(x)}\right)$$

find $f'(5)$

If we know how to derive for ln,

$$f'(x) = \left[\ln\left(\frac{3g(x)}{h(x)}\right)^2\right]' = \left[2\ln\left(\frac{3g(x)}{h(x)}\right)\right]'$$

$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

$\ln ab = \ln a + \ln b$

$$= 2[\ln 3g(x) - \ln h(x)]'$$

$$= 2[\ln 3 + \ln g(x) - \ln h(x)]'$$

$D[\ln 3] = 0$, since $\ln 3$ is a constant

$$f'(x) = 2\left[\frac{1}{g(x)} \cdot g'(x) - \frac{1}{h(x)} \cdot h'(x)\right]$$

$$f'(5) = 2\left[\frac{1}{g(5)} \cdot g'(5) - \frac{1}{h(5)} \cdot h'(5)\right]$$

$$= 2\left[\frac{1}{-3} \cdot 6 - \frac{1}{3} \cdot -2\right]$$

$$= -\frac{8}{3}$$

3.5 Implicit Differentiation

Definition 3.5.1.

In the equation $y = 3x^2 - 5$, the variable y is explicitly written as a function of x. Some functions, however, are only implied by an equation. For instance, the function $x^2 - 2y^3 + 4y = 2$ is defined implicitly. To find $\frac{dy}{dx}$, you can use implicit differentiation.

Note 3.5.2 Guidelines for Implicit Differentiation

1) Differentiate both sides of the equation in terms of x
2) Collect all terms involving $\frac{dy}{dx}$ on the left side and the other terms to the right side of the equation
3) Factor out $\frac{dy}{dx}$ on the left side
4) Solve for $\frac{dy}{dx}$

Example 3.5.3

1. Find $\frac{d}{dx}[xy^2]$.

$$= x \cdot [y^2]' + x' \cdot y^2$$

$$= x \cdot 2y\left(\frac{dy}{dx}\right) + y^2 \quad \text{or} \quad x \cdot 2y \cdot y' + y^2$$