Derivative of a Function

Def. The tangent line to the curve $y = f(x)$ at the point $(a, f(a))$ is the line through $(a, f(a))$ with slope $m = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$.

Ex. $y = f(x) = x^2$

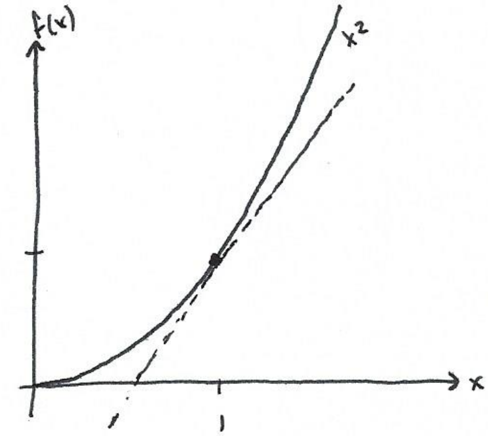
$(1,1)$ is on graph of f

The slope of the dashed line (tangent line) at $(1,1)$ is given by

$$m = \lim_{x \to 1} \frac{f(x) - f(1)}{x-1}$$

$$= \lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1} \frac{(x-1)(x+1)}{x-1}$$

$$= \lim_{x \to 1} x+1 = 2$$



slope @ $(0,0)$ of tangent line: $m = \lim_{x \to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0} \frac{x^2-0}{x} = \lim_{x \to 0} x = 0$

@ arbitrary $(a, f(a))$: $m = \lim_{x \to a} \frac{f(x)-f(a)}{x-a} = \lim_{x \to a} \frac{x^2-a^2}{x-a} = \lim_{x \to a} \frac{(x-a)(x+a)}{x-a} = \lim_{x \to a} x+a = a+a = 2a$

$\rightarrow m = 2a$

Def. The derivative of $y = f(x)$ at a is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x-a}$$

Ex. Find $f'(a)$ for $f(x) = x^2 - 8x + 9$

$a=3$: $f'(3) = \lim_{x \to 3} \frac{f(x)-f(3)}{x-3} = \lim_{x \to 3} \frac{(x^2-8x+9)-(-6)}{x-3} = \lim_{x \to 3} \frac{x^2-8x+15}{x-3} = \lim_{x \to 3} \frac{(x-3)(x-5)}{x-3} = \lim_{x \to 3} x-5 = -2$

$f(3) = 9-24+9$
$= -6$

in general: $f'(a) = \lim_{x \to a} \frac{1}{x-a}\left[x^2-8x+9-(a^2-8a+9)\right] = \lim_{x \to a} \frac{1}{x-a}\left[x^2-8x-a^2+8a\right]$

$$= \lim_{x \to a} \frac{1}{x-a}\left[x^2-a^2-(8x-8a)\right] = \lim_{x \to a} \cancel{\frac{1}{x-a}}\left[\cancel{(x-a)}(x+a) - 8\cancel{(x-a)}\right]$$

$$= \lim_{x \to a} \left[x+a-8\right] = a+a-8 = 2a-8$$