January 29, 2019

Check Blackboard for unit circle.

2.4 Continuity and one-sided limits

- A function $f$ is continuous at $c$ when the following are true:
  1. $f(c)$ is defined
  2. $\lim_{x \to c} f(x)$ exists
  3. $\lim_{x \to c} f(x) = f(c)$

Example: Let $y = x^2$ such that it is graphed below.

![Graph of $y = x^2$]

If $c = 1$, what is true?

1. $f(c)$ is defined
   $f(1) = 2$, \text{ } \checkmark$

2. $\lim_{x \to c} f(x)$ exists
   $\lim_{x \to 1} x^2 = 1$, \text{ } \checkmark$

3. $\lim_{x \to c} f(x) = f(c)$
   \text{ } \checkmark$

But

$f(1) = 2$, \text{ } \checkmark$
A function is **continuous** on an open interval \((a, b)\) when the function is continuous at each point in the interval.

A function that is continuous on the entire real number line \((-\infty, \infty)\) is everywhere continuous.

If a function \(f\) is defined on \(I\) (except possibly \(c\)) and \(f\) is not continuous at \(c\), then \(f\) is said to have a discontinuity at \(c\).

Discontinuities fall in two categories.

- **Removable** A discontinuity at \(c\) is called removable when \(f\) can be made continuous by appropriately defining \(f\) at \(c\). Otherwise, \(c\) is \(\text{non-removable}\).
The left and right limits are the same value. They approach the discontinuity. If we put a point at \( x = c \), the graph would be continuous.

This discontinuity at \( c \) is removable.

Similar situation as above. The limit exists, and putting a point at \( x = c \) would complete the graph as continuous.

At \( x = c \), there is a removable discontinuity.

The left and right limits are different values. So the limit DNE. There is also a break (jump) between the two points on \( c \).

So at \( x = c \), there is a non-removable discontinuity.
Another way to differentiate between removable and non-removable discontinuities...

1. If the function factors and the bottom term cancels, the discontinuity at the x-value for which the denominator was zero is removable, so the graph has a hole in it.

   \[ g(x) = \frac{x^2 - 1}{x - 1} \]

   \[ x = 1 \text{ gives division by 0.} \]

   \[ g(x) = \frac{(x+1)(x-1)}{x-1} \]

   \[ \text{factor and cancel.} \]

   \[ g(x) = x + 1. \]

   There is a removable discontinuity at \( x = 1 \).

2. If a term does not cancel, the discontinuity at this x-value corresponding to this term for which the denominator is zero is non-removable, and the graph has a vertical asymptote.

   \[ f(x) = \frac{1}{x} \]

   \[ x = 0 \text{ gives division by 0, but we cannot transform this to anything.} \]

   There is a non-removable discontinuity at \( x = 0 \) (vertical asymptote.)
Examples Are the following continuous?

\[ h(x) = \begin{cases} 
  x + 1 & \text{if } x \leq 0 \\
  e^x & \text{if } x > 0 
\end{cases} \]

We use this formula if \( x \) is less than or equal to 0.

We use this formula if \( x \) is greater than 0.

\[ \text{If } x \leq 0 \]

\[ h(x) = x + 1 \]

We can add 1 to any \( x \)-value and still get a real number. So all values of \( x \) equal to and less than 0 are defined.

\[ \text{If } x > 0 \]

\[ h(x) = e^x \]

\( e \) will be any real number if \( x \) is positive. So all values of \( x \) greater than 0 are defined.

Does the limit exist at \( x = 0 \)?

Left of 0 is the negatives and we use \( x + 1 \). We call this the left-hand limit.

\[ \lim \limits_{x \to 0^-} h(x) = \lim \limits_{x \to 0^-} x + 1 = 0 + 1 = 1 \]

Right of 0 is the positives and we use \( e^x \). This is the right-hand limit.

\[ \lim \limits_{x \to 0^+} h(x) = \lim \limits_{x \to 0^+} e^x = e^0 = 1 \]

The left-hand and right-hand limits are both 1. Therefore the limit exists.

\[ \lim \limits_{x \to 0} h(x) = 1 \]