January 24, 2019

2.3 Evaluating Limits Analytically

THEOREM Let a, c be real numbers and let n be a positive integer.

Then $\lim_{x \to c} a = a$ $\quad \lim_{x \to c} x = c$ $\quad \lim_{x \to c} x^n = c^n$

Examples

1. $\lim_{x \to 2} 3 = 3$

   ($\lim_{x \to c} a = a$)

2. $\lim_{x \to -4} x = -4$

   ($\lim_{x \to c} x = c$)

3. $\lim_{x \to 2} x^3 = 8$

   ($\lim_{x \to c} x^n = c^n$)

THEOREM Let a, c be real numbers n be a positive integer and let f, g be functions with $\lim_{x \to c} f(x) = L$, $\lim_{x \to c} g(x) = K$. Then

① Scalar multiple | $a \lim_{x \to c} f(x) = \lim_{x \to c} a f(x) = aL$

② Sum / Difference | $\lim_{x \to c} (f(x) \pm g(x)) = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x) = L \pm K$

③ Product | $\lim_{x \to c} (f(x) g(x)) = \lim_{x \to c} f(x) = \lim_{x \to c} g(x) = LK$

THEOREM cont.

④ Quotient $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \frac{L}{K}, K \neq 0$

⑤ Power $\lim_{x \to c} (f(x))^n = \left(\lim_{x \to c} f(x)\right)^n = L^n$

Example $\lim_{x \to 2} (4x^2 + 3) = ?$

$= \lim_{x \to 2} 4x^2 + \lim_{x \to 2} 3$ using the ② sum/diff. property

$= 4 \lim_{x \to 2} x^2 + \lim_{x \to 2} 3$ using the ① scalar mult. property

$= 4 \cdot 2^2 + \lim_{x \to 2} 3$ using the ⑤ power property

$= 4 \cdot 2^2 + 3$

$\boxed{\lim_{x \to 2} (4x^2 + 3) = 19}$

THEOREM Limits of Polynomial and Rational Functions

If $P(x)$ is a polynomial function and c is a real number $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$

Then $\lim_{x \to c} P(x) = P(c)$.

If $r(x) = \frac{p(x)}{q(x)}$ is a rational function and c is a real number such that $q(c) \neq 0$, then $\lim_{x \to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}$

Let n be a positive integer.
Then $\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$.

THEOREM cont.

If $f(x)$ and $g(x)$ are functions such that $\lim_{x \to c} g(x) = L$ and $\lim_{x \to c} f(x) = f(L)$,

Then $\lim_{x \to c} f(g(x)) = f(\lim_{x \to c} g(x))$

$= f(L)$

Example

1. If $\lim_{x \to c} f(x) = 3$ and $\lim_{x \to c} g(x) = 2$

$\lim_{x \to c} [5f(x)g(x)]^2$

$= [\lim_{x \to c} 5f(x)g(x)]^2$ power property

$= [5\lim_{x \to c} f(x)g(x)]^2$ scalar mult. property

$= [5(\lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x))]^2$

$= [5 \cdot 3 \cdot 2]^2 = [30]^2$

$\boxed{\lim_{x \to c} [5f(x)g(x)]^2 = 900}$

2. If $f(x) = 5 - x$, $g(x) = x^3$. Find $\lim_{x \to 1} g(f(x))$.

$\lim_{x \to 1} g(f(x)) = \lim_{x \to 1} g(5-x)$

$= \lim_{x \to 1} (5-x)^3$

$= (5-1)^3 = 4^3$

$\boxed{\lim g(f(x)) = 64}$

THEOREM Let c be a real number in the domain of the trignometric function.

Then

$$\lim_{x \to c} \sin x = \sin c \qquad \lim_{x \to c} \csc x = \csc c$$

$$\lim_{x \to c} \cos x = \cos c \qquad \lim_{x \to c} \sec x = \sec c$$

$$\lim_{x \to c} \tan x = \tan c \qquad \lim_{x \to c} \cot x = \cot c$$

Example

1. $\lim_{x \to e} \ln(x^3) = \ln(e^3)$
$= 3 \ln e$

$$\boxed{\lim_{x \to e} \ln(x^3) = 3}$$

2. $\lim_{x \to \frac{\pi}{6}} \cos x \cdot \sin^2 x = \cos\frac{\pi}{6} \cdot \left(\sin\frac{\pi}{6}\right)^2$
$= \frac{\sqrt{3}}{2} \cdot \left(\frac{1}{2}\right)^2$

$$\boxed{\lim_{x \to \frac{\pi}{6}} \cos x \cdot \sin^2 x = \frac{\sqrt{3}}{8}}$$

3. $\lim_{x \to 1} \frac{x^3 + 2x^2 - 1}{5 - 3x} = \frac{1 + 2 - 1}{5 - 3} = \frac{2}{2}$

$$\boxed{\lim_{x \to 1} \frac{x^3 + 2x^2 - 1}{5 - 3x} = 1}$$

4. $\lim_{x \to 1} \frac{x^2 - x}{x^2 + 3x - 4} = \lim_{x \to 1} \frac{x\cancel{(x-1)}}{(x+4)\cancel{(x-1)}} = \lim_{x \to 1} \frac{x}{x+4} = \frac{1}{1+4}$

$$\boxed{\lim_{x \to 1} \frac{x^2 - x}{x^2 + 3x - 4} = 5}$$

Reminder

$\ln(x^a) = a \ln x$

$\ln(xy) = \ln x + \ln y$

$\ln \frac{x}{y} = \ln x - \ln y$

$\log_c c = 1$

$(\ln x)^a \neq a \ln x$