One-sided limits — same as before, but we treat $x>a$ and $x<a$ separately.

\[
\lim_{x \to a^+} f(x) = L \\
\lim_{x \to a^-} f(x) = L
\]

with $x>a$, we can make $f(x)$ arbitrarily close to $L$ by making $x$ sufficiently close to $a$.

with $x<a$, we can make $f(x)$ as close to $L$ as we like by making $x$ sufficiently close to $a$.

**Heaviside function**

\[
H(t) = \begin{cases} 
0, & t < 0 \\
1, & t \geq 0
\end{cases}
\]

\[
\lim_{t \to 0^-} H(t) = 0 \
\lim_{t \to 0^+} H(t) = 1
\]

\[
\text{Prop.} \quad \lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L.
\]
Example \( f(x) = \frac{1}{x^2} \)

**Domain?** - all real numbers except 0

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>-1</th>
<th>-\frac{1}{2}</th>
<th>-\frac{1}{4}</th>
<th>\frac{1}{2}</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>( \frac{1}{4} )</td>
<td>1</td>
<td>4</td>
<td>16</td>
<td>4</td>
<td>1</td>
<td>( \frac{1}{4} )</td>
</tr>
</tbody>
</table>

\[ \lim_{x \to 0} f(x) = \infty, \text{ so the limit DNE} \]

**Continuity**

**Def.** The function \( f \) is **continuous at the number** \( a \) means:

1. \( f(a) \) exists (\( a \) is in domain of \( f \))
2. \( \lim_{x \to a} f(x) \) exists
3. \( \lim_{x \to a} f(x) = f(a) \)

**Ex.** \( H(t) \)

\[ H(t) = \begin{cases} 0, & t \leq 0 \\ 1, & t > 0 \end{cases} \]

\( H(t) \) is continuous at each \( a \neq 0 \).

\( H(t) \) is not continuous at \( t = 0 \) because

\[ \lim_{t \to 0^-} H(t) \neq \lim_{t \to 0^+} H(t), \text{ so } \lim_{t \to 0} H(t) \text{ DNE} \]
Ex. \( f(x) = \frac{1}{x^2} \)

- \( f \) is not continuous at \( x = 0 \) because \( f(x) \) does not exist.
- \( f \) is continuous at each \( a \neq 0 \), so \( f \) is continuous on its domain.

**Functions Continuous on their domains:**

- **Polynomials:** \( a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = P(x) \)
- **Rational functions:** \( \frac{P(x)}{Q(x)} \)
- **Power, root functions:** \( x^n, x^{1/n} \)
- **Trig functions:** \( \sin(x), \tan(x) \)
- **Exponential, logarithmic functions:** \( e^x, \log(x) \)
- **Products, quotients, sums, differences, compositions of above**

Ex. \[
\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x+1)(x-1)}{(x-1)} = \lim_{x \to 1} x + 1 = 1 + 1 = 2
\]

Ex. Evaluate the limit

\[
\lim_{x \to -4} \frac{x + \frac{1}{x}}{4 + x} = \lim_{x \to -4} \frac{x + \frac{4}{4x}}{4 + x} = \lim_{x \to -4} \frac{1}{4x} = \frac{-1}{16}
\]