Section 4.4

Gram-schmidt process

Given, a set of linearly independent vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in $\mathbb{R}^n$ (which span some subspace of $\mathbb{R}^n$), we wish to construct a set of k orthonormal vectors $\hat{u}_1, u_2, \ldots u_k$ which span the same subspace of $\mathbb{R}^n$.

1 vector

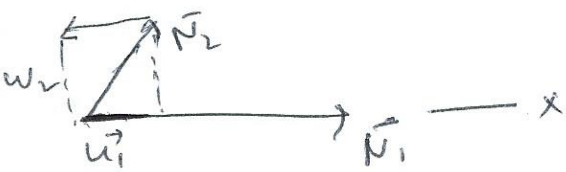
$\vec{v}_1 \in \mathbb{R}^n$

To find $\vec{u}_1$:

Let $\bar{v}_1 = \bar{w}_1 \Rightarrow \boxed{\vec{u}_1 = \frac{\bar{w}_1}{\|\bar{w}_1\|}}$

2 vectors,

$\vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$



To find $\hat{u}_1$:

Let $\vec{v}_1 = \vec{w}_1 \Rightarrow \hat{u}_1 = \frac{w_1}{\|w_1\|}$

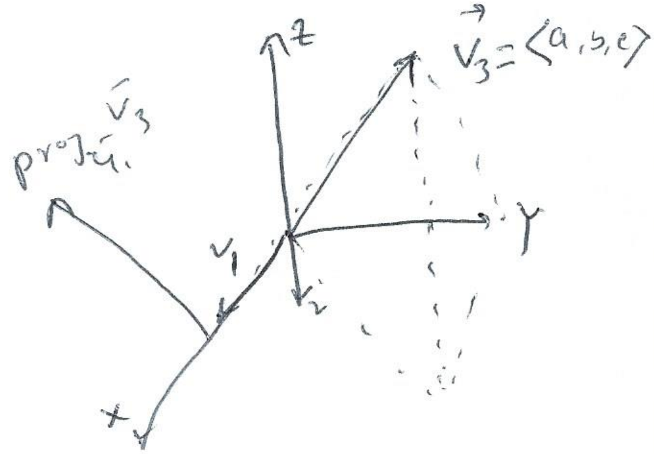
To find $\vec{u}_2$:

Let $w_2 = \vec{v}_2 - \mathrm{proj}_{\bar{u}_1} \bar{v}_2$

$\Rightarrow \hat{u}_2 = \frac{\bar{w}_2}{\|\bar{w}_2\|}$

3 vectors

$\bar{v}_1, \bar{v}_2, \bar{v}_3 \in \mathbb{R}^n$



To find $\vec{u}_1$:

Let $\bar{w}_1 = \bar{v} \Rightarrow \hat{u}_1 = \frac{w_1}{\|\bar{w}_1\|}$

To find $\vec{u}_2$ let $\bar{w}_2 = \vec{v}_2 - \mathrm{proj}\, v_2$

$\Rightarrow \vec{u}_2 = \frac{w_2}{\|w_2\|}$

To find $\vec{u_3}$:

$$\bar{w_3} = v_3 - proj_{u_1} v_3 - proj_{u_2} v_3$$

$$\Rightarrow \quad \bar{u_3} = \frac{w_3}{\|w_3\|}$$

k vectors

$v_1, v_2, \ldots v_n \in \mathbb{R}^n$

u_1: let, $w_1 = v_1 \Rightarrow \hat{u_1} = \frac{w_1}{\|w_1\|}$

u_2: let, $w_2 = v_1 - proj_{u_1} v_2 \Rightarrow \hat{u_2} = \frac{w_2}{\|w_2\|}$

Apply the gram-schmidth process to the vectors

$$\vec{v_1} = (1, 1, 1, 1) \quad \vec{v_2} = (0, 1, 1, 1), \quad v_3 = (0, 0, 1, 1)$$

$$\vec{u_1} = \text{let } w_1 = v_1 = (1, 1, 1, 1) \Rightarrow \vec{u_1} = \frac{w_1}{\|w_1\|}$$

$$\|w_1\| = \|v_1\| = \sqrt{1^2+1^2+1^2+1^2} = \sqrt{4} = 2$$

$$\Rightarrow \vec{u_1} = \frac{1}{2}\bar{w_1} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = \vec{u_1}$$

$\vec{u_2}$: let $w_2 = v_2 - proj_{u_1} v_2 = (0, 1, 1, 1) - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} \hat{u_1}$

$$= (0, 1, 1, 1) - \left(0 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2}\right) \hat{u}$$

$$= (0, 1, 1, 1) - \tfrac{3}{2}\left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right)$$

$$= (0, 1, 1, 1) - \left(\tfrac{3}{4}, \tfrac{3}{4}, \tfrac{3}{4}, \tfrac{3}{4}\right)$$

$$w_2 = \left(-\tfrac{3}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}\right)$$

$$\Rightarrow \vec{u_2} = \frac{w_2}{\|w_2\|}$$

$$||w_2|| = \sqrt{(-3/4)^2 + (\tfrac{1}{4})^2 + (\tfrac{1}{4})^2 + (\tfrac{1}{4})^2}$$

$$= \sqrt{\frac{9+1+1+1}{16}}$$

$$= \sqrt{\frac{12}{16}} = \frac{\sqrt{3}}{2} \left(-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{3}}\right)$$

$$= \vec{u}_2$$

u_3:

let $\bar{w}_3 = \bar{v}_3 - \text{proj}_{u_1} v_3 - \text{proj}_{u_2} v_3$

$$w_3 = (0,0,1,1) - (v_3 \cdot u_1)\vec{u}_1 - (v_3 \cdot u_2)\vec{u}_2$$

$$= (0,0,1,1) - (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) - (-\tfrac{1}{2}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6})$$

$$v_3 \cdot u_1 = 0 + 0 + \tfrac{1}{2} + \tfrac{1}{2}$$

$$\vec{v}_3 \cdot \vec{u}_2 = 0 + 0 + \frac{1}{2\sqrt{3}} + \frac{1}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$$