4.4

Orthogonal Bases

Given two vectors \( \mathbf{u}, \mathbf{v} \)

\[
\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}
\]

in \( \mathbb{R}^n \),

the dot product of \( \mathbf{u}, \mathbf{v} \) is:

\[
\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n
\]

and the following properties hold:

1. \( \mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} \)
2. \((\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}
3. \((c \mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = (\mathbf{u} \cdot c \mathbf{v})
4. \( \mathbf{u} \cdot \mathbf{u} > 0 \), and \( \mathbf{u} \cdot \mathbf{v} = 0 \) only when \( \mathbf{u} = \mathbf{0} \).

Note: \( \mathbf{u} \) and \( \mathbf{v} \) are perpendicular iff \( \mathbf{u} \cdot \mathbf{v} = 0 \).

Definition:

A collection of vectors in \( \mathbb{R}^n \) is called orthogonal if any two are perpendicular.

⇒ A set of vectors \( \{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \} \) in \( \mathbb{R}^n \) is called orthogonal if \( \mathbf{v}_i \cdot \mathbf{v}_j = 0 \) for all \( \mathbf{v}_i, \mathbf{v}_j \) with \( i \neq j \).

Example:

Show that \( \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} \) is an orthogonal set of vectors, while

\[
\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}
\]
$v_1 \cdot v_2 = 0 \quad v_1 \cdot v_3 = 0 \quad v_2 \cdot v_3 = 0$

$11 + (-4) + (-7) = 3 + (-8) + 5 = 33 + (12) + (-33) = 0 = 0 = 0$

**Theorem:**
If $v_1, \ldots, v_k$ are orthogonal (all non-zero), then they are linearly independent.

**Proof:**
Suppose $a_1, \ldots, a_k \in \mathbb{R}$ such that

$$a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + \cdots + a_k \mathbf{v}_k = \mathbf{0}$$

want to show that

$$a_1 = a_2 = \cdots = a_k = 0$$

Choose any $i$ such that $1 \leq i \leq k$.

Then

$$(a_1 \mathbf{v}_1 + \cdots + a_k \mathbf{v}_k) \cdot \mathbf{v}_i = \mathbf{0} \cdot \mathbf{v}_i$$

**Def:**
A vector in called normal if $\|v\| = 1$

A collection of vectors in $\mathbb{R}^n$

$v_1, \ldots, v_k \in \mathbb{R}^n$ in called orthonormal if they are orthogonal and each $\|v_i\| = 1$.

An orthonormal basis in a basis that is made up of orthonormal vectors.
Ex.
The standard basis for $\mathbb{R}^n$
\[ \{ e_1, e_2, \ldots, e_n \} \]
in an orthonormal basis.

\[ V_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad V_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad V_3 = \begin{bmatrix} 3 \\ -2 \\ -5 \end{bmatrix} \]

we have shown these are orthogonal $\Rightarrow$ linearly independent $\Rightarrow$ 3 linearly independent vectors in $\mathbb{R}^3$

$\Rightarrow$ in a basis for $\mathbb{R}^3$

Normalizing
\[ \bar{U}_1 = \frac{\bar{V}_1}{\| V_1 \|} \quad \bar{U}_2 = \frac{\bar{V}_2}{\| V_2 \|} \quad \bar{U}_3 = \frac{\bar{V}_3}{\| V_3 \|} \]

$\Rightarrow \{ \bar{U}_1, \bar{U}_2, \bar{U}_3 \}$ in an orthonormal basis for $\mathbb{R}^3$. 