4.4

Orthogonal Bases

Given, two vector $\vec{u},\vec{v}$
$$\vec{u}=\begin{bmatrix}u_1\\ \vdots\\ u_n\end{bmatrix},\quad \vec{v}=\begin{bmatrix}v_1\\ \vdots\\ v_n\end{bmatrix} \text{ in } \mathbb{R}^n,$$

the dot. product of $\vec{u}$ & $\vec{v}$ is:

$$\boxed{\vec{u}\cdot\vec{v}=u_1v_1+u_2v_2+\cdots+u_nv_n}$$

and the following properties hold:

1. $\vec{u}\cdot\vec{v}=\vec{v}\cdot u$
2. $(u+v)\cdot w=u\cdot w+v\cdot w$
3. $(c\vec{u})\cdot\vec{v}=u\cdot(c\vec{v})=c(\vec{u}\cdot\vec{v})$
4. $\vec{u}\cdot\vec{u}\geq 0$, and $\vec{u}\cdot\vec{v}=0$ only when $\vec{u}=\vec{0}$.

u and v are perpendicular iff $\vec{u}\cdot\vec{v}=0$

Def:

A collection of vectors in $\mathbb{R}_n$ is called orthogonal if any two are perpendicular.

$\Rightarrow$ A set of vector $\{v_1,v_2,\cdots v_k\}$ in $\mathbb{R}^n$ is called orthogonal ~~set of~~ if $\bar{v}_i\cdot\bar{v}_j=0$ for all v_i, v_j with $i\neq j$.

Ex:
Show that $\{v_1,v_2,v_3\}$ is an orthogonal set of vectors,

while
$$v_1=\begin{bmatrix}1\\4\\-1\end{bmatrix}\quad v_2=\begin{bmatrix}11\\-1\\7\end{bmatrix}\quad v_3=\begin{bmatrix}3\\-2\\-5\end{bmatrix}$$

check ?

$v_1 \cdot v_2 = 0$ $v_1 \cdot v_3 = 0$ $v_2 \cdot v_3 = 0$

$= 11 + (-4) + (-7)$ $= 3 + (-8) + 5$ $= 33 + (+2) + (-35)$

$= 0$ $= 0$ $= 0$

Theorem :

if $v_1, \ldots v_K$ are orthogonal (all non-zero), then they are linearly independent.

proof :

suppose $a_1, \ldots a_K \in \mathbb{R}$ such that

$a_1 \vec{v_1} + a_2 \vec{v_2} + \cdots + a_K \vec{v_K} = \vec{0}$ → want to show that

$a_1 = a_2 = \cdots = a_K = 0$

choose any i such that $1 \leq i \leq K$:

then

$(a_1 \vec{v_1} + \cdots + a_K \vec{v_K}) \cdot v_i = \vec{0} \cdot \vec{v_i}$

Def :

A vector is called normal if $\|\vec{v}\| = 1$

A collection of vectors in $\mathbb{R}^n$

$v_1, \ldots v_k \in \mathbb{R}^n$ is called orthonormal

if they are orthogonal and each $\|\vec{v_i}\| = 1$.

An orthonormal basis is a basis that is made up of orthonormal vectors.

Ex.

The standard basin for $\mathbb{R}^n$

$\{ \varepsilon_1, \varepsilon_3, \varepsilon_n \}$ in an orthonormal basin.

$$v_1 = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 11 \\ -1 \\ 7 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 3 \\ -2 \\ -5 \end{bmatrix}$$

we have shown these are orthogonal $\Rightarrow$ linearly independent $\Rightarrow$ 3 linearly independent vectors in $\mathbb{R}^3$

$\Rightarrow$ in a basin for $\mathbb{R}^3$

Normalize

$$\bar{u}_1 = \frac{\bar{v}_1}{\|\bar{v}_1\|} \qquad \bar{u}_2 = \frac{\bar{v}_2}{\|\bar{v}_2\|} \qquad \bar{u}_3 = \frac{\bar{v}_3}{\|\bar{v}_3\|}$$

$\Rightarrow \{u_1, u_2, u_3\}$ in an orthonormal basin for $\mathbb{R}^3$.