linear algebra

①

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3.6 basis and dimensions

Def:

A basis for a vector space V is a set S such that

① S is linearly independent

② span{S} = V

plural of the basis is bases.

Theorem: if a vector space V has a basis S with n number of element, then any other basis for V also has n elements.

Def:

let V be a vector space:

① The dimension of V is n, if any basis for V has n elements.
↳ every

② The dimension of the zero vector space is zero.
(The zero vector space has no basis)

$V = \{\vec{0}\} \Rightarrow$ No basis $\Rightarrow$ Dimension zero.

$V = \mathbb{R}^n \Rightarrow$ mem bases $\Rightarrow$ Dimension $\Rightarrow n$

Example:

let, $S = \{\varepsilon_1, \varepsilon_2, \cdots \varepsilon_n\}$

where, $\varepsilon_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\varepsilon_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$; $\cdots$ $\varepsilon_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

S is a basis for $\mathbb{R}^n$:

1) S is linearly independent? is the only soln to

$c_1 \varepsilon_1 + c_2 \varepsilon_2 + \cdots + c_n \varepsilon_n = \vec{0}$, $c_1 = c_2 = \cdots = c_n = 0$?

Let, $c_1, c_2, c_n \in \mathbb{R}$ such that

$$c_1 \varepsilon_1 + \cdots + c_n \varepsilon_n = \vec{0}$$

$$c_1\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix} + \cdots c_n\begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix}$$

$$\begin{bmatrix}c_1\\\vdots\\c_n\end{bmatrix} = \begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix} \Rightarrow c_1, \cdots, c_n = 0$$

2) Does S span $\mathbb{R}^n$? is $\text{span}\{S\} = \mathbb{R}^n$?

is the set of all linear combination of elements of S.

if $\bar{x} \in \mathbb{R}^n$, are there $c_1, \dots c_n \in \mathbb{R}$

$$\bar{x} = c_1 \varepsilon_1 + c_2 \varepsilon_2 + \cdots c_n \varepsilon_n ?$$

Let, $\bar{x} \in \mathbb{R}^n$, then:

$$\bar{x} = (x_1, x_2, \cdots x_n) = x_1(1, 0, 0, \cdots 0) + x_2(0, 1, 0, \cdots 0) + x_n(0, 0, 0, \cdots n)$$

$$\bar{x} = x_1 \varepsilon_1 + x_2 \varepsilon_2 + \cdots + x_n \varepsilon_n$$

$S = \{\varepsilon_1, \cdots \varepsilon_n\}$ is called "standard basis" for $\mathbb{R}^n$

Let, $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, where

$$\vec{v}_1 = \begin{bmatrix}2\\1\\0\end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix}-3\\-3\\1\end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix}-2\\1\\-1\end{bmatrix}$$

show that, S is a basis for $\mathbb{R}^3$

Show, L is linearly independent and span $\mathbb{R}^3$.

Does S span $\mathbb{R}^3$?

if $\vec{x} \in \mathbb{R}^3$, are there number $c_1, c_2, c_3 \in \mathbb{R}$

such that, $c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{x}$?

$\Rightarrow$ Does the system

$$\begin{bmatrix}2 & -3 & -2\\1 & -3 & 1\\0 & 1 & -1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$ have a solution?

Augmented matrix

$$\left[\begin{array}{ccc|c}2 & -3 & -2 & x_1\\1 & -3 & 1 & x_2\\0 & 1 & -1 & x_3\end{array}\right] \xrightarrow{R_2: -\frac{1}{2}R_1 + R_2} \left[\begin{array}{ccc|c}2 & -3 & -2 & x_1\\0 & -3/2 & 2 & -1/2 x_1 + x_2\\0 & 1 & -1 & x_3\end{array}\right] \rightarrow$$

$$\xrightarrow{R_3: \frac{2}{3}R_2 + R_3} \left[\begin{array}{ccc|c}2 & -3 & -2 & x_1\\0 & -3/2 & 2 & -\frac{1}{2}x_1 + x_2\\0 & 0 & \frac{1}{3} & -\frac{1}{3}x_1 + \frac{2}{3}x_2 + x_3\end{array}\right]$$

This has a solution for x_1, x_2, x_3 : span $\mathbb{R}^3$

Choose $\vec{x} = \vec{0}$

The system becomes

$$\Rightarrow \left[\begin{array}{ccc|c}2 & -3 & -2 & 0\\0 & -3/2 & 2 & 0\\0 & 0 & 1/3 & 0\end{array}\right] \Rightarrow \text{only}$$

trivial soln $\Rightarrow$ S is linearly independent