3.4

Null spaces:

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Def:

Let A be an $m \times n$ matrix,

then the null space of A is a subspace of $\mathbb{R}^n$, denoted $NS(A)$.

$$NS(A) = \{ \bar{x} \in \mathbb{R}^n \mid A\bar{x} = \bar{0} \}$$

* The null space of A is the solution set of homogeneous system. $A\bar{x} = \bar{0}$

$\boxed{Ax = b \text{ matrix form}}$

$\boxed{Ax = b^T \text{ vector form}}$

Examples:

Find $NS(A)$ for the following

$A_1 = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

$A_1\vec{x} = \bar{0}$

$\Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} \ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 + 2x_2 = 0$

$3x_2 = 0$

$\boxed{x_2 = 0}$ $\boxed{x_1 = 0}$

$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

null space $= \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

$A_2 = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

$A_2 \vec{x} = \vec{0}$

$\Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 6 & 0 \end{array}\right] \xrightarrow{R_2 : -3R_1 + R_2}$

$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$x + 2x_2 = 0$

x_2 → free variable

$x_1 = -2x_2$

let, $t = x_2$

$x = -2t$

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2t \\ t \end{bmatrix} \quad t \in \mathbb{R}$

$\vec{x} = t\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$$NS(A) = \left\{ \vec{x} \in \mathbb{R}^2 \mid \bar{x} = t\begin{bmatrix} -2 \\ 1 \end{bmatrix}, t \in \mathbb{R} \right\}$$

$A_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$A_3\vec{x} = \vec{0}$

$\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$0x_1 + 0x_2 = 0$ true for all $x_1, x_2 \in \mathbb{R}$

So, $\boxed{NS(A) = \mathbb{R}^2}$

Def:

A vector $\vec{w}$ is a linear combination of the vector $\vec{v}_1, \vec{v}_2, \vec{v}_n$

if there are numbers a_1, a_2, a_n such that.

$$a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = \vec{w}.$$

Fact: if $\vec{w}$ is a linear combination of $\vec{v}_1, \ldots, \vec{v}_n$, that representation might not be unique.

Reminder:

if A is an $m\times n$ matrix and $\bar{x}$ is an $n\times 1$ vector, then the product $A\vec{x} = \vec{b}$ is the linear combination of the columns of A.

Ex:

let $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$

Determine if either

$\vec{w}_1 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ or $w_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ is a linear combination of $\vec{v}_1, \vec{v}_2$

Need to determine:

Are the numbers A_1, A_2 such that $a_1\vec{v}_1 + a_2\vec{v}_2 = \vec{w}_1$

$$\begin{bmatrix} \vdots & \vdots \\ v_1 & v_2 \\ \vdots & \vdots \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}}$$

are there any soln?

$$\left[\begin{array}{cc:c} 2 & -4 & 3 \\ 1 & -2 & 6 \end{array}\right]$$

$$\hookrightarrow \left[\begin{array}{cc:c} 2 & -4 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

$2a_1 - 4a_2 = 3$

$a_1 = \frac{3}{2} + 2a_2$

IS w_1 a linear combination of r_1, r_2?

yes!

Represent w_1 as a linear combition of $\vec{r}_1, \vec{r}_2$.

$$w_1 = a_1 r_1 + a_2 r_2$$

$$\begin{bmatrix} 3 \\ 6 \end{bmatrix} = (2)\begin{bmatrix} 2 \\ 1 \end{bmatrix} + (0)\begin{bmatrix} -4 \\ -2 \end{bmatrix}$$

* are the $a_1, a_2 \in \mathbb{R}$ such that $a_1 \vec{r_1} + a_2 \vec{r_2} = \vec{w}_2$

Does

$$\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$ have solutions?

$$\left[\begin{array}{cc|c} 2 & -4 & 3 \\ 1 & -2 & 5 \end{array}\right] \xrightarrow{R_2 : -\frac{1}{2}R_1 + R_2} \left[\begin{array}{cc|c} 2 & -4 & 3 \\ 0 & 0 & 7/2 \end{array}\right]$$ no solution.

$\Rightarrow w_2$ is not a linear combination of r_1 & r_2

Def:

Let $\vec{r_1} \cdots \vec{r_n}$ be vectors in a vector space V. The set of all linear combinations of $\vec{v}_1, \cdots \vec{v}_n$ in ~~what~~ called the span of $\vec{r}_1, \cdots \vec{r}_n$. denoted by $\text{span}\{r_1, \cdots r_n\}$.

and we say that $\vec{v}_1 \cdots \vec{v}_n$ span S.

Let, $\varepsilon_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \hat{i}$

$\varepsilon_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \hat{j}$

$\varepsilon_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \hat{k}$

$$\text{span}\{\varepsilon_1, \varepsilon_2, \varepsilon_3\} = \mathbb{R}^3$$

$\Rightarrow$ Any vector in $\mathbb{R}^3$ is a linear combination of $\varepsilon_1, \varepsilon_2, \varepsilon_3$

let, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$

$$\vec{x} = x_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad x_1, x_2, x_3 \in \mathbb{R}$$

$$= \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$