Ex.

Find a basis for the subspace of $\mathbb{R}^5$ spanned by

$$\vec{v}_1 = \begin{bmatrix} 2 \\ -3 \\ 4 \\ 1 \\ 2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 4 \\ -6 \\ 5 \\ 3 \\ -4 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} -10 \\ 15 \\ -14 \\ 5 \\ 7 \end{bmatrix} \quad v_4 = \begin{bmatrix} -8 \\ 12 \\ -10 \\ 4 \\ 3 \end{bmatrix}$$

Set the vectors as the row of the matrix and find a basis for the row space.

$$A = \begin{bmatrix} 2 & -3 & 4 & 1 & 2 \\ 4 & -6 & 5 & 3 & -4 \\ -10 & 15 & -14 & 3 & 7 \\ -8 & 12 & -10 & 4 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & -3 & 4 & 1 & 2 \\ 0 & 0 & -3 & 1 & -8 \\ 0 & 0 & 0 & 10 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$= u$

A basis for $RS(A)$ is:

$RS(u)$

$$\left\{ \begin{bmatrix} 2 \\ -3 \\ 4 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -3 \\ 1 \\ -8 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 10 \\ 1 \end{bmatrix} \right\}$$

Given an $m \times n$ matrix A:

if A can be reduced to row echelon form u:

$NS(A) = NS(u)$

$\dim[NS(A)] = \#$ of free variables in u

$RS(A) = RS(u)$

$\dim[RS[A]] = \#$ of leading variable in u

$CS(A) \neq CS(u)$

column space

when A reduced to U, a basis for $CS(U)$ is the set of column vectors which contain the pivots of U.

A ~~basis~~ basis for $CS(A)$ is produced by choosing the column vectors of A which correspond to the columns of U containing the pivots.

$$\text{Basis for } CS(U) = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 10 \\ 1 \end{bmatrix} \right\}$$

$$\text{Basis for } CS(A) = \left\{ \begin{bmatrix} 2 \\ 4 \\ -10 \\ -8 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ -14 \\ -10 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 3 \\ 4 \end{bmatrix} \right\}$$

Rank of A, denoted Rank(A), is the $\dim[RS(A)] =$

$\dim[CS(A)] = \#$ of pivots in U.

$\text{rank}(A) + \dim[NS(A)] = n$

rank-nullity theorem

Ex: find a basis for $RS(A)$, $CS(A)$, $NS(A)$ and verify

$\text{rank}(A) + \dim[NS(A)] = n$

$$A = \begin{bmatrix} 3 & -6 & -4 & 1 & 5 \\ 6 & 4 & -7 & 3 & 1 \\ -3 & -2 & 6 & 1 & 2 \\ 3 & 6 & -1 & 4 & 6 \end{bmatrix} \xrightarrow{\text{row operation}} \begin{bmatrix} \boxed{3} & 2 & -4 & 1 & 5 \\ 0 & 0 & \boxed{1} & 1 & -9 \\ 0 & 0 & 0 & 0 & \boxed{25} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = U$$

$(m \times n) = (4 \times 5)$

$$\text{Basis for } RS(A) = \left\{ \begin{bmatrix} 3 \\ 2 \\ -4 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 25 \end{bmatrix} \right\}$$

Basin for $CS(A) = \left\{ \begin{bmatrix} 3 \\ 6 \\ -3 \\ 9 \end{bmatrix}, \begin{bmatrix} -4 \\ -7 \\ 6 \\ -11 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 2 \\ 6 \end{bmatrix} \right\}$

$NS(A)$: find solution to $\bar{u}x = 0$

$NS(A)$:

$$\left[\begin{array}{ccccc|c} 3 & 2 & -4 & 1 & 5 & 0 \\ 0 & 0 & 1 & 1 & -9 & 0 \\ 0 & 0 & 0 & 0 & 25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\uparrow$ $\uparrow$

free variables

$$3x_1 + 2x_2 - 4x_3 + x_4 + 5x_5 = 0$$

$$x_3 + x_4 - 9x_5 = 0$$

$$25x_5 = 0$$

$$x_5 = 0$$

Let $x_2 = t_1$

$x_4 = t_2$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$x_1 = -\frac{2}{3}t_1 - \frac{5}{3}t_2$

$\boxed{x_3 = -t_2}$ $\boxed{x_4 = t_2}$

$\boxed{x_5 = 0}$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3}t_1 - \frac{5}{3}t_2 \\ t_1 + 0 \\ 0 - t_2 \\ 0 + t_2 \\ 0 + 0 \end{bmatrix}$$

$$= t_1 \begin{bmatrix} -5/3 \\ 0 \\ -t_2 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -5/3 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

So, a basis for $NS(U) = NS(A) = \left\{ \begin{bmatrix} -2/3 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -5/3 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right.$