Let, $A = \begin{bmatrix} -2 & 1 & 3 \\ 4 & -2 & 5 \end{bmatrix}$

Note that

1) The columns of A span $\mathbb{R}^2$
2) The rows of A are linearly independent

1) The column of A can be "contracted" to form a basis for $\mathbb{R}^2$.

2) The row of A can be "expanded" to form a basis for $\mathbb{R}^3$.

1) $c_1 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$ $c_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ $c_3 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

Need to ~~at~~ take away a vector, so that the ~~linear~~ remaining ones are linearly independent.

$c_1 = -2c_2 \to \{c_1, c_2\}$ is linearly dependent:

$c_1 \neq k c_3$ for any $k \in \mathbb{R}$

$c_2 \neq r c_3$ for any $r \in \mathbb{R}$

So either $\{c_1, c_2\}$ or $\{c_2, c_3\}$ is LI

So either of those sets form a basis for $\mathbb{R}^2$.

2. $r_1 = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$ $r_2 = \begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix}$

Need to find another vector r_3 in $\mathbb{R}^3$. Such that $\{r_1, r_2, r_3\}$ spans $\mathbb{R}^3$.

standard basis

$\{i, j, k\}$

$\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$

Make a "good guess" with $\varepsilon_1, \varepsilon_2, \ldots$

- in this case, add $\hat{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ or $\hat{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

So, use either $\{r_1, r_2, \hat{i}\}$ or $\{r_1, r_2, \hat{j}\}$

cannot use $\tilde{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ because:

$$2r_1 + r_2 = \begin{bmatrix} -4 \\ 2 \\ 6 \end{bmatrix} + \begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 11 \end{bmatrix} = 11\hat{k}$$

$\boxed{3.7}$

fundamental subspace of a matrix:

$NS(A)$, $RS(A)$, $CS(A)$

Let A be an $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Row vectors of A:

$$r_1 = [a_{11} \; a_{12} \; \cdots \; a_{1n}]$$

$$\vdots$$

$$r_m = [a_{m1} \; a_{m2} \; \cdots \; a_{mn}]$$

column vectors of A:

$$c_1 = \begin{bmatrix} a_{11} \\ a_{12} \\ a_{1n} \end{bmatrix} \; \cdots \; c_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \Bigg\} \Rightarrow \text{in } \mathbb{R}^m$$

Def:

1) The subspace of $\mathbb{R}^m$ spanned by row vectors of A is called row space of A denoted $RS(A)$

2) The subspace of $\mathbb{R}^m$ spanned by the column vectors of A is called the column space denoted $CS(A)$

Remember:

$NS(A)$ is a subspace of $\mathbb{R}^m$.

- we have learned how to find a basis and dimension of $NS(A)$
- we want to do the same for $RS(A)$!
- To find basis for $NS(A)$, we used the fact that $NS(A) = NS(u)$. [u is echelon form of A]

Finding a basis for $RS(A)$:

- if A is $m \times n$ and can be reduced to u (row echelon) then $RS(A) = RS(u)$.

Find a basis for $RS(A)$

$$A = \begin{bmatrix} 2 & -3 & 4 & 1 & 2 \\ 4 & -6 & 5 & 3 & -4 \\ -10 & 15 & -14 & 3 & 7 \\ 8 & 12 & -10 & 4 & 9 \end{bmatrix} \xrightarrow[\text{operations}]{\text{row}} \begin{bmatrix} 2 & -3 & 4 & 1 & 2 \\ 0 & 0 & -3 & 1 & -8 \\ 0 & 0 & 0 & 10 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = u$$

Fact:
Let U be a matrix in row echelon form, then the non-zero rows of U form a basis for $RS(U)$

Use a non-zero rows as a basis for $RS(U)$.

$\Rightarrow$ A basis for $RS(A) = RS(U)$ is:

$$\left\{ \begin{bmatrix} 2 \\ -3 \\ 4 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -3 \\ 1 \\ \cancel{3}\,-8 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Fact.
Dimension of $RS(A)$ is equal to the number of leading variable in U.