Linear algebra

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3.6

Basis and dimension

if V is a vector space, and S is a set of vectors in V,

then S is a basis for V if:

1. S is linearly independent
2. span V. (span{S} = V)

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every basis V has the same number of element, n, called the dimension of V.

Basis and Dimension of NS(A). [A is m×n]

$$NS(A) = \{ \vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0} \}$$

- NS(A) is a subspace of $\mathbb{R}^n$

Find a basis for NS(A):

$$A = \begin{bmatrix} 0 & 0 & 1 & -4 \\ 2 & -4 & -1 & -2 \\ 4 & -8 & 0 & -12 \end{bmatrix}$$

1) Reduce A to echelon form U.
2) use the fact that NS(A) = NS(U)
3) Find basis for NS(U) = NS(A)

$$A = \begin{bmatrix} 0 & 0 & 1 & -4 \\ 2 & -4 & -1 & -2 \\ 4 & -8 & 0 & -12 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 2 & -4 & -1 & -2 \\ 0 & 0 & 1 & -4 \\ 4 & -8 & 0 & -2 \end{bmatrix}$$

$$\xrightarrow{R_3 : -2R_1 + R_3} \begin{bmatrix} 2 & -4 & -1 & -2 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 2 & -8 \end{bmatrix}$$

$$\xrightarrow{R_3 : -2R_2 + R_3} \begin{bmatrix} 2 & -4 & -1 & -2 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

$2x_1 - 4x_2 - x_3 - 2x_4$

$NS(U)$:

solutions to

$U\vec{x} = \vec{0}$

$$\left[\begin{array}{cccc|c} 2 & -4 & -1 & -2 & 0 \\ 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left.\begin{array}{l} \boxed{2x_1} - 4x_2 - \boxed{x_3} - 2x_4 = 0 \\ \qquad\qquad\quad \boxed{x_3} - 4x_4 = 0 \end{array}\right\} \Rightarrow \begin{array}{l} 2x_1 - x_3 = 4x_2 - 2x_4 \\ x_3 = 4x_4 \end{array}$$

↑ ↑

leading variables

let $x_2 = t_1$ and $x_4 = t_2$

$2x_1 - x_3$ and $x_4 = t_2$

$x_3 = 4t_2$

$2x_1 - x_3 = 4t_1 + 2t_2$

$2x_1 - 4t_2 = 4t_1 + 2t_2$

$2x_1 = 4t_1 + 6t_2$

$x_1 = 2t_1 + 3t_2$

if $\vec{x} \in NS(U) = NS(A)$, then it is of the form:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2t_1 + 3t_2 \\ t_1 + 0t_2 \\ 0t_1 + 4t_2 \\ 0t_1 + t_2 \end{bmatrix} = \begin{bmatrix} 2t_1 \\ t_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 3t_2 \\ 0 \\ 4t_2 \\ t_2 \end{bmatrix}$$

$$= t_1 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 3 \\ 0 \\ 4 \\ 1 \end{bmatrix}$$

$t_1, t_2 \in \mathbb{R}$

so a basis NS(U) = NS(A) is

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 4 \\ 1 \end{bmatrix} \right\}$$

what is dimension of NS(A) ?

$\dim(NS(A)) = 2$.

important fact: Fact :

if U is in echelon form, then

$\dim[NS(U)]$ = the number of free variables

$\dim[NS(U)]$ = the number of free variable in the system.

$$U\vec{x} = \vec{0}$$

let V be a vector space ($V \neq \{\vec{0}\}$)

and S be a set of vectors in V

$$S = \{ v_1, v_2, \cdots v_t \}$$

Then

1) if S spans V, then either S is the basis for V, or a finite number of vectors can be ~~added~~ removed from S to form a basis.

2. if S is linearly independent, then either S is a basis for V, or finite number of vectors can be added to S to form a basis.

Theorem 3.64 page 193.

Ex. let $A = \begin{bmatrix} -2 & 1 & 3 \\ 4 & -2 & 5 \end{bmatrix}$

$\Rightarrow$ the column of A span $\mathbb{R}^2$

$v_1 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$ $\quad u_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ $\quad v_3 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$