Linear algebra 02-04-2019 (1) Lec → 6

→ consider the linear eqn.

$$a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots \qquad\qquad \vdots$$
$$a_{m1}x_1 + \cancel{a_m}\ a_{m2}x_2 + \cdots a_{mn}x_n = b_m$$

Let $A = \underbrace{\begin{bmatrix} a_{11} & a_{1n} \cdots & a_{1n} \\ a_{21} & & \vdots \\ \vdots & & \vdots \\ a_{m1} & a_{n2} \cdots & a_{mn} \end{bmatrix}}_{\text{matrix coefficient}}$ $\bar{X} = \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{\text{variable matrix}}$ $B = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{\text{matrix of constant}}$

Associated matrix equation:

$$A\bar{X} = B$$

$$A\bar{X} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \cdots x_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 \cdots a_{2n}x_n \\ a_{m1}x_1 + a_{m2}x_2 \cdots a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = B$$

$(m\times n)(n\times 1) \Rightarrow (m\times 1)$

$\boxed{Ex}$

Find the associated matrix eqn for:

$$\boxed{\begin{aligned} 2x - 3y + 4z &= 5 \\ 4x \qquad - z &= -2 \end{aligned}}$$

suppose variables x, y, z ↳ depend on some other variables, say t_1, t_2

$$\overset{A}{\begin{bmatrix} 2 & -3 & 4 \\ 4 & 0 & -1 \end{bmatrix}} \overset{\bar{X}}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

→ Produce a new expression of original system

$$x = 3t_1 - t_2$$
$$y = 2t_1 + 5t_2$$
$$z = -t_1 + 2t_2$$

~~x~~ $3t_1 - t_2 = x$

$2t_1 + 5t_2 = y$

$-t_1 + 2t_2 = z$

Matrix Eq

$$\begin{bmatrix} 3 & -1 \\ 2 & 5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

old substitution method

$$2(t_1 - t_2) - 3(2t_1 + 5t_2) + 4(-t_1 + 2t_2) = 5$$

$$4(3t_1 - t_2) - (-t_1 + 2t_2) = -2$$

simplify

$$\boxed{\begin{aligned} -4t_1 - 9t_2 &= 5 \\ 13t_1 - 6t_2 &= -2 \end{aligned}}$$

using matrix multiplication

$$\begin{bmatrix} 2 & -3 & 4 \\ 4 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -3 & 4 \\ 4 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 3 & -1 \\ 2 & 5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} \right) = \cancel{\begin{bmatrix} 5 \\ -2 \end{bmatrix}} \begin{bmatrix} -4 & -9 \\ 13 & -6 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

$AX = B$

$A(CT) = B$ $\quad$ ~~$(ACT = B$~~ ~~$D\,D$~~ $\quad$ $(AC)T = B$

~~B~~ $DT = B$

For $AX = B$; the product AX is a linear combination" of the column of A.

$$A\underline{\bar{X}} = \begin{bmatrix} a_{11}x_1 \\ a_{21}x_1 \\ a_{m1}x_1 \end{bmatrix} + \begin{bmatrix} a_{12}x_2 \\ a_{22}x_2 \\ \vdots \\ a_{m2}x_2 \end{bmatrix} + \cdots + \begin{bmatrix} a_{1n}x_n \\ a_{2n}x_n \\ \vdots \\ a_{mn}x_n \end{bmatrix}$$

$$= x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{1n} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$\begin{bmatrix} 2 & -3 & 4 \\ 4 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

check that $(-1, -5, -2)$ is a solution.

$$A \qquad x = B$$

$$-1\begin{bmatrix} 2 \\ 4 \end{bmatrix} + -5\begin{bmatrix} -3 \\ 0 \end{bmatrix} + -2\begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

Let A be an $m \times n$ (square matrix)

An inverse of A is an $m \times n$ matrix A^{-1} with the property:

$$AA^{-1} = I = A^{-1}A$$

where $I = I_n$

Fact: not every $m \times n$ matrix has an inverse

Ex: $A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$. Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ 0 & 0 \end{bmatrix} \neq I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow$ A has no inverse

if a matrix has an inverse, it is unique.

Theorem:

if A and B are invertible ~~n x n~~ $n \times n$ then (AB) is also invertible,

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$(AB)(AB^{-1}) = A(BB^{-1})A^{-1} = A(I_n)A^{-1} = AA^{-1} = I_n$$

$$(AB)^{-1}(AB) = B^{-1}(A^{-1}A)B = B^{-1}(I_n)B = B^{-1}B = I_n$$

Inverses of 2×2 matrices:

let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\boxed{A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

$$AA^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

~~$A^{-1}A$~~

$A^{-1}A =$