3.1 : 3.2 Vector Spaces

A vector in Euclidean n-space is an ordered n-tuple of real numbers \( \mathbf{u} = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n \) denoted \( \mathbb{R}^n \).

Vector operations:

If \( \mathbf{u}, \mathbf{v} \) are vectors in \( \mathbb{R}^n \) and \( c \) is a scalar (real number), then

\[
\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)
\]

\[
c \mathbf{u} = (cu_1, cu_2, \ldots, cu_n)
\]

Examples:

Let \( \mathbf{u} = (1, 2, 3, 4) \) and \( \mathbf{v} = (4, 3, 2, 1) \) in \( \mathbb{R}^4 \)

\[
c = \frac{1}{2}
\]

\[
\mathbf{u} + \mathbf{v} = (1+4, 2+3, 3+2, 4+1) = (5, 5, 5, 5)
\]

\[
c \mathbf{u} = \left( \frac{1}{2}, 1, \frac{1}{2}, 2 \right)
\]
\[ \vec{u} = (u_1, u_2, \ldots, u_n) \]

Direction:

\[ \overrightarrow{PQ} = (x_2 - x_1, y_2 - y_1) \]

Magnitude:

\[ \|\vec{u}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]

Properties of vectors:

1. Let \( \vec{u}, \vec{v}, \vec{w} \) be vectors, \( r, s \) be scalars,
   \[ \vec{u} + \vec{v} = \vec{v} + \vec{u} \]
   \[ (\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w}) \]
   \[ \vec{u} + (-\vec{u}) = \vec{0} \]
   \[ r(s \vec{u}) = (rs) \vec{u} \]

2. The additive inverse of \( \vec{u} \) is \( -\vec{u} \), such that
   \[ \vec{u} + (-\vec{u}) = \vec{0} \]

3. The magnitude of \( \vec{u} \) in the distance between start/terminal point:
   \[ \|\vec{u}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]
2 Dimension

Unusually, we let the starting point be the origin: (0, 0, 0)

\[ \mathbf{u}^1 = (x, y) \]

components

\[ \mathbf{u}^1, \mathbf{u}^2 = (2, 1), \quad \mathbf{v}^3 = (1, 3), \quad \mathbf{u}^1 + \mathbf{v}^3 = (3, 4) \]

Scalar multiplication:

\[ \mathbf{u}^1(2, 1) \rightarrow \text{same direction as } \mathbf{u}^1, \text{ twice as long} \]

\[ 2 \mathbf{u}^1 = (4, 2) \]

\[ \frac{1}{2} \mathbf{u}^1 = (1, \frac{1}{2}) \rightarrow \text{same direction as } \mathbf{u}^1, \text{ but half as long} \]

\[ -\mathbf{u}^1 = (-2, -1) \rightarrow \text{same size as } \mathbf{u}^1, \text{ in opposite direction} \]