partial pivoting (1.7)

consider the system (imagine you are a computer)

- round ~~it to~~ everything to 8 sig. digit

① $-\ 10^{-8}x + y = 1$

② $-\ x + y = 2$

- A Solution : $(x, y) = (1, 1)$

① $-\ 10^{-8}(1) + 1 = .00000001 + 1$

$= 1.00000001$

⑪ $1 + 1 = 2$

Do, Gaussian elimination :

$$\left[\begin{array}{cc|c} 10^{-8} & 1 & 1 \\ 1 & 1 & 2 \end{array}\right]$$

$-10^{8}R_1 + R_2$

R3 : $\longrightarrow \left[\begin{array}{cc|c} 10^{-8} & 1 & 1 \\ 0 & -10^{8} & -10^{8} \end{array}\right]$

$-10^{8}(10^{-8}) = -1$

$$\begin{array}{l} 10^{-8}x + y = 1 \\ -10^{8}x + -10^{8}y = 2(-10^{8}) \\ \hline 0 - 10^{8}y = -10^{8} \end{array}$$

$\Rightarrow y = 1 \Rightarrow 10^{-8}x + 1 = 1$

$\Rightarrow x = 0$

But, $(x, y) = (0, 1)$ is not a solution.

if a number being zero will cause problems mathematically, then that number near zero will cause problems numerically.

$$-10^{-8}(x+y=2) \qquad\qquad -10^{-8}x + -.00000001y = -.00000002$$

$$10^{-8}x+y=1 \qquad\qquad 10^{-8}x + y \qquad\qquad y=1$$

$$0$$

$$y - .00000001y = 1 - .00000002$$

$$.99999999y = .99999998$$

$$y = 1$$

let, $A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 16 & 64 \\ 2 & 4 & 8 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 4 & 16 & 64 \\ 1 & 1 & 1 \\ 2 & 4 & 8 \end{bmatrix} \xrightarrow[R_2:\ -\frac{1}{4}R_1+R_2]{R_3:\ -\frac{1}{2}R_1+R_2} \begin{bmatrix} 4 & 16 & 64 \\ 0 & -3 & -15 \\ 0 & -4 & -24 \end{bmatrix}$

use partial pivoting to solve using

$$PA = LU$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 4 & 16 & 64 \\ 0 & -4 & -24 \\ 0 & -3 & -15 \end{bmatrix} \xrightarrow{R_3:\ -\frac{3}{4}R_2+R_3} \begin{bmatrix} 4 & 16 & 64 \\ 0 & -4 & -24 \\ 0 & 0 & 3 \end{bmatrix} = U$$

$$P \to \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = P$$

$$\begin{matrix} 4 & 16 & 64 \\ 1 & 1 & 1 \\ 2 & 4 & 8 \end{matrix} \qquad PA = \begin{bmatrix} 4 & 16 & 64 \\ 2 & 4 & 8 \\ 1 & 1 & 1 \end{bmatrix}$$

$$PA = \begin{bmatrix} 4 & 16 & 64 \\ 2 & 4 & 8 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow[R_2:\ -\frac{1}{2}R_1+R_2]{R_3:\ -\frac{1}{4}R_1+R_3} \begin{bmatrix} 4 & 16 & 64 \\ 0 & -4 & -24 \\ 0 & -3 & -15 \end{bmatrix} \longrightarrow$$

$R3: -\frac{3}{4}R_2+R_3$ (3)

$$\xrightarrow{\qquad} \begin{bmatrix} 4 & 16 & 64 \\ 0 & -4 & -24 \\ 0 & 0 & 3 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{4} & 3/4 & 1 \end{bmatrix}$$

$\boxed{\text{Solve}}$

$AX = B$

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 16 & 64 \\ 2 & 4 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 100 \\ 6 \end{bmatrix}$$

$AX = B$

$PAX = PB = B'$

$$B' = PB = \begin{bmatrix} 100 \\ 6 \\ -2 \end{bmatrix}$$

$(LU)x = B'$

$L(Ux) = B'$

let $y = Ux$, solve $Ly = B'$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/4 & 3/4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 100 \\ 6 \\ -2 \end{bmatrix} \Rightarrow \quad y_1 = 100$$

$\frac{1}{2}(100) + y_2 = 6$

$\boxed{y_2 = -44}$

$$y = \begin{bmatrix} 100 \\ -44 \\ 6 \end{bmatrix}$$

$\frac{1}{4}(100) + 3/4(-44) + y_3 = -2$

$y_3 = 6$

Then solve $Ux = y$ for x

$$\begin{bmatrix} 4 & 16 & 64 \\ 0 & -4 & -24 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 100 \\ -44 \\ 6 \end{bmatrix} \Rightarrow x_3 = 2, \ x_2 = -1, \ x_1 = -3$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}$$