Feb 13, 2019

LU decomposition with row interchange ($PA = LU$)

If $A$ is non-singular (invertible), then there is a permutation matrix $P$ such that $\boxed{PA = LU}$ → unique, only after finding a specific $P$

- A permutation matrix is a product of elementary permutation matrices.

Process:

Start with $A$

- Do gaussian elimination on $A$ to determine all row interchanges to get $P$

- Do gaussian elimination again to determine $LU$

Simple example: (2x2)

$$A = \begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix} = U$$

$$P = \begin{bmatrix} \end{bmatrix}$$

Elementary matrix to swap $R_1$ & $R_2$

Start with

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = P_{12}$$

$PA = LU$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix} = \underset{L}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}\underset{U}{\begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}}$$

Ex:

$A = \begin{bmatrix} 0 & 0 & 3 \\ 2 & 4 & -1 \\ 6 & 5 & 5 \end{bmatrix}$ find a $PA = LU$ decomposition.

$$A = \begin{bmatrix} 0 & 0 & 3 \\ 2 & 4 & -1 \\ 6 & 5 & 5 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 2 & 4 & -1 \\ 0 & 0 & 3 \\ 6 & 5 & 5 \end{bmatrix} \xrightarrow{R_3 : -3R_1 + R_3} \begin{bmatrix} 2 & 4 & -1 \\ 0 & 0 & 3 \\ 0 & -7 & 8 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 2 & 4 & -1 \\ 0 & -7 & 8 \\ 0 & 0 & 3 \end{bmatrix} = U$$

Determine $P$:

Start with idendity matrix, apply row interchange to get $P$.

$$I_n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} [\quad] \to \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = P$$

To find $L$:

$$PA = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 3 \\ 2 & 4 & -1 \\ 6 & 5 & 5 \end{bmatrix} = [\quad$$

$$A = \begin{bmatrix} 0 & 0 & 3 \\ 2 & 4 & -1 \\ 6 & 5 & 5 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 2 & 4 & -1 \\ 0 & 0 & 3 \\ 6 & 5 & 5 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 2 & 4 & -1 \\ 6 & 5 & 5 \\ 0 & 0 & 3 \end{bmatrix} = PA$$

$$\begin{bmatrix} 2 & 4 & -1 \\ 6 & 5 & 5 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow{R_2 : -3R_1 + R_2} \begin{bmatrix} 2 & 4 & -1 \\ 0 & -7 & 8 \\ 0 & 0 & 3 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(3)

$PA = LU$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 3 \\ 2 & 4 & -1 \\ 6 & 5 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 & -1 \\ 0 & -7 & 8 \\ 0 & 0 & 3 \end{bmatrix}$$

Solve,

$Ax = B$ where $A = \begin{bmatrix} 0 & 0 & 3 \\ 2 & 4 & -1 \\ 6 & 5 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 6 \\ -10 \\ -7 \end{bmatrix}$

$A\cancel{x = B}$

1st → find $PA = LU$

$PAx = B$

$PAx = PB = B'$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 6 \\ -10 \\ -7 \end{bmatrix} = \begin{bmatrix} -10 \\ -7 \\ 6 \end{bmatrix} = B'$$

$PAx = B'$

$LU(x) = B'$

$L(Ux) = B'$

let $y = Ux$ $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Solve $Ly = B'$

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -10 \\ -7 \\ 6 \end{bmatrix}$$

forward substitution

$y_1 = -10 \Rightarrow 3(-10) + y_2 = -7$

$y_2 = 23$

$y_3 = 6$

$PAx = B'$

$(LU)x = B'$

$L(Ux) = B'$

let $Y = Ux$ Solve for $Ly = B'$ for $Y$, then solve $Ux = y$ for $x$.

$Ux = y$

$$\begin{bmatrix} 2 & 4 & -1 \\ 0 & -7 & 8 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -10 \\ 23 \\ 6 \end{bmatrix}$$

Back sub:

$3x_3 = 6$

$\boxed{x_3 = 3}$

$\Rightarrow -7x_2 + 8(2) = 23$

$\boxed{x_2 = \cancel{0}\ -1}$

$\Rightarrow 2x_1 + 4(-1) - 2 = -10 \to$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$$