1.3 Matrix algebra

An $m \times n$ matrix is a rectangular array of m rows and n columns.

(m, n: positive integers) of the form:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = A = [a_{ij}]$$

The number a_{ij} are called the entries.

m and n are called dimensions.

Two matrices A and B is equal if $a_{ij} = b_{ij}$ for all i

Equality of matrices

Ex: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \ln(1) & e^0 \\ 2-1 & 2-2 \end{bmatrix}$

Sum of matrices

- the matrices must have the same dimensions

$A + B = [a_{ij} + b_{ij}]$

Ex: let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$; $B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

both 2×3 matrices

$$A + B = \begin{bmatrix} 1+0 & 2+1 & 3+0 \\ 3+1 & 2+0 & 1+1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 3 \\ 4 & 2 & 2 \end{bmatrix}$$

$A + B = B + A$

$A + (B + C) = (A + B) + C$

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$m \times n$ zero matrix:

- An $m \times n$ matrix whose entries are all zero.
- additional identity

if A is $m \times n$ then $A + O_{m \times n} = A = O_{mn} + A$

Scalar multiplication:

Let a be a scalar (real number).

Let B be an $m \times n$ matrix

$a \times B = [a b_{ij}]$

Ex: Let $B = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}$

the $3B = \begin{bmatrix} 3(2) & 3 \times 0 & 3 \times 1 \\ 3(1) & 3 \times 2 & 3 \times 0 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 3 \\ 3 & 6 & 0 \end{bmatrix}$

and $-1B = \begin{bmatrix} -2 & 0 & -1 \\ -1 & -2 & 0 \end{bmatrix} = -B$ ~ additive inverse

$\Rightarrow B + -B = 0$

Sustraction:

$A - B = A + (-1) B$

Multiplying matrices:

if A in $m \times n$, and B in $n \times P$

$C = AB$ in $m \times P$

and the elements of the product C are given by:

$$c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj}$$

wrong:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1(5) & 2(6) \\ 3(7) & 4(8) \end{bmatrix}$$

Ex:

Let $A \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}$; $B = \begin{bmatrix} -1 & 0 & 2 \\ 4 & -3 & -1 \end{bmatrix}$

$[2 \times 2]$ $\qquad$ $[2 \times 3]$ $\rightarrow$ result should be 2×3

$$AB = \begin{bmatrix} 3(-1) + 1(4) & 3(0) + 1(-3) & 3(2) + 1(-1) \\ -2(-1) + 0(4) & (-2)(0) + 0(-3) & (-2 \times 2) + 0(-1) \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & -3 & 5 \\ 2 & 0 & -4 \end{bmatrix}$$

Let $A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$ $\quad B = \begin{bmatrix} 4 & -2 \\ -1 & 1 \end{bmatrix}$

AB should be 2×2

$$AB = \begin{bmatrix} 2 \times 4 + (-1)(-1) & 2(2) + (-1)(1) \\ 1 \times 4 + 3(-1) & 1(-2) + 3(1) \end{bmatrix} = \begin{bmatrix} 9 & -5 \\ 1 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 4(2) + (-2)(1) & 4(-1) + (-2)(3) \\ -1(2) + (1)(1) & -1(-1) + 1(3) \end{bmatrix} = \begin{bmatrix} 6 & -10 \\ -1 & 4 \end{bmatrix}$$

in general $AB \neq BA$

in fact AB might be defined.

while BA is not

$BA \rightarrow [2 \times 3][2 \times 2]$ not defined

$A(BC) = (AB)C$ as long as all the products are defined.

$A(B+C) = AB + AC$, $(A+B)C = AC + BC$

if A is $m \times n$ and B is $n \times p$,

then $A I_n = A$ and $I_n B = B$

where, $I_n = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Ex. $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$