More gaussian elimination :

Find all solution of the linear system

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 2 \end{array}\right]$$

inconsistent

$$\begin{aligned} 2x_1 - 3x_2 + 10x_3 &= -2 \\ x - 2x_2 + 3x_3 &= -2 \\ -x_1 + 3x_2 + x_3 &= 4 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 2 & -3 & 10 & -2 \\ 1 & -2 & 3 & -2 \\ -1 & 3 & 1 & 4 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -2 & 3 & -2 \\ 2 & -3 & 10 & -2 \\ -1 & 3 & 1 & -4 \end{array}\right]$$

$$\xrightarrow[R_2 : -2R_1 + R_2]{R_3 : R_3 + 1R_1} \left[\begin{array}{ccc|c} 1 & -2 & 3 & -2 \\ 0 & 1 & 4 & 2 \\ 0 & 1 & 4 & 2 \end{array}\right]$$

$$\xrightarrow{R_3 : -1R_2 + R_3} \left[\begin{array}{ccc|c} 1 & -2 & 3 & -2 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

leading variable, free variable

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= -2 \\ x_2 + 4x_3 &= 2 \end{aligned}$$

move the free variable to the right hand side

$$\begin{aligned} x - 2x_2 &= -2 - 3x_3 \\ x_2 &= 2 - 4x_3 \end{aligned}$$

Let, $\boxed{x_3 = t}$, where $t \in \mathbb{R}$

$$\left.\begin{aligned} x_1 - 2x_2 &= -2 - 3t \\ x_2 &= 2 - 4t \end{aligned}\right\} \Rightarrow \boxed{x_2 = 2 - 4t}$$

$$\Rightarrow x_1 - 2(2 - 4t) = -2 - 3t$$

$$x_1 - 4 + 8t = -2 - 3t$$

$$\boxed{x_1 = 2 - 11t}$$

Solution are of the form:

$(x_1, x_2, x_3) = (2-11t, 2-4t, t)$ where $t \in \mathbb{R}$

Some specific solution:

$t=0 \rightarrow (2, 2, 0)$
$t=1 \rightarrow (-9, -2, 1)$

Find all solutions of linear system

$$\begin{aligned} 2x_1 - 6x_2 - x_3 + 8x_4 &= 0 \\ x_1 - 3x_2 - x_3 + 6x_4 &= 0 \\ -x_1 + 3x_2 - x_3 + 2x_4 &= 0 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 2 & -6 & -1 & 8 & 0 \\ 1 & -3 & -1 & 6 & 0 \\ -1 & 3 & -1 & 2 & 0 \end{array}\right]$$

Homogeneous system

- all equations of the form
$a_1x_1 + \cdots + a_nx_n = 0$
- always have at least one solution: $x_1 = x_2 = x_3 = x_4 = 0$
(trivial solution)

$R_1 \leftrightarrow R_2$

$$\left[\begin{array}{cccc|c} 1 & -3 & -1 & 6 & 0 \\ 2 & -6 & -1 & 8 & 0 \\ -1 & 3 & -1 & 2 & 0 \end{array}\right] \xrightarrow[R_3 : R_1 + R_3]{R_2 : -2R_1 + R_2} \left[\begin{array}{cccc|c} 1 & -3 & -1 & 6 & 0 \\ 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & -2 & 8 & 0 \end{array}\right] \xrightarrow{R_3 = 2R_2 + R_3}$$

$$\left[\begin{array}{cccc|c} 1 & -3 & -1 & 6 & 0 \\ 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Leading variable (with pivots) — free variable

$$\begin{aligned} x_1 - 3x_2 - x_3 + 6x_4 &= 0 \\ x_3 - 4x_4 &= 0 \\ \hline x_1 - x_3 &= 3x_2 - 6x_4 \\ x_3 &= 4x_4 \end{aligned}$$

Let, $x_2 = t_1$, $x_4 = t_2 \Rightarrow x_3 = 4t_2$

$\Rightarrow x_1 - 4t_2 = 3t_1 - t_2$

$x_1 = 3t_1 - 2t_2$

Solutions are of the form

$(x_1, x_2, x_3, x_4) = (3t_1 - 2t_2, t_1, 4t_2, t_2)$ where $t_1, t_2 \in \mathbb{R}$