Ex. Find constants $a, b$ so that $f(x)$ is continuous on $\mathbb{R}$,

\[
\begin{align*}
    f(x) &= \begin{cases} 
        -2 & x \leq -1 \\
        ax + b & -1 < x < 3 \\
        -2 & x \geq 3 
    \end{cases} 
\end{align*}
\]

* need to check continuity @ $x=-1, 3$

So we need 

\[
\begin{align*}
    \lim_{x \to -1^-} f(x) &= f(-1) \\
    \lim_{x \to -1^+} f(x) &= f(-1) \\
    \lim_{x \to 3^-} f(x) &= f(3) \\
    \lim_{x \to 3^+} f(x) &= f(3)
\end{align*}
\]

\[
\begin{align*}
    \lim_{x \to -1} f(x) &= f(-1) = 2 \\
    \lim_{x \to 3} f(x) &= f(3) = -2
\end{align*}
\]

\[
\begin{align*}
    \lim_{x \to -1^+} (ax + b) &= -a + b = 2 \\
    \lim_{x \to 3^-} (ax + b) &= 3a + b = -2
\end{align*}
\]

\[
\begin{align*}
    b &= a + 2 = (1) + 2 = 1 \\
    a + 2 &= 4a + 2 = 4(-1) + 2 = -2
\end{align*}
\]

So 

\[
\begin{align*}
    f(x) &= \begin{cases} 
        2, & x \leq -1 \\
        1-x, & -1 < x < 3 \\
        -2, & x \geq 3
    \end{cases}
\end{align*}
\]

**Intermediate Value Theorem**

If $f$ is continuous on $[a,b]$, $f(a) \neq f(b)$, and $f(c) \leq k < f(b)$, then there is a number $c$ in $[a,b]$ such that $f(c) = k$.

Ex. Show that $f(x) = x^3 + 2x - 1$ has a zero on $[0,1]$.

\[
\begin{align*}
    f(0) = 0^3 + 2(0) - 1 &= -1 \\
    f(1) = 1^3 + 2(1) - 1 &= 2
\end{align*}
\]

$f(x)$ is a polynomial 

$f$ is continuous 

$0 \in (-1,2)$, so by IVT there exists $c \in [0,1]$ such that $f(c) = 0$.
Infinite Limits

An infinite limit is a limit in which \( f(x) \) increases or decreases without bound as \( x \) approaches \( c \): \( \lim_{x \to c} f(x) = \infty \).

\( \lim_{x \to c} f(x) = \infty \) does not mean that the limit exists. It tells you how the limit fails to exist by denoting the unbounded behavior of \( f(x) \) as \( x \) approaches \( c \).

\[ \frac{1}{x} \]

\[ \lim_{x \to 1^-} g(x) = -\infty \]
\[ \lim_{x \to 1^+} g(x) = \infty \]

The line \( x = c \) is a vertical asymptote if at least one of the following is true:

\[ \lim_{x \to c} f(x) = \pm \infty \]
\[ \lim_{x \to c^+} f(x) = \pm \infty \]
\[ \lim_{x \to c^-} f(x) = \pm \infty \]

Ex: Find any vertical asymptotes.

1) \( g(x) = \cot(x) = \frac{\cos(x)}{\sin(x)} \)
   \[ \sin(x) = 0 \]
   \( x = k\pi, \ k \in \mathbb{Z} \) (vertical asymptotes here)

2) \( h(x) = x^2 e^{-x} = \frac{x}{e^x} \)
   \( e^x = 0 \)
   \( x = \ln(0) = \text{DNE} \)
   \( x^2 - 4 = 0 \)
   \( x = \pm \sqrt{4} = \pm 2 \)

3) \( s(x) = \ln(x^2 - 4) \)
   no vertical asymptotes
Since we can factorize the numerator to cancel out the zero in the denominator at \(x = 5\), this point is a removable discontinuity, not a vertical asymptote.

Then let \(c\) and \(L\) be real numbers and let \(f\) and \(g\) be functions such that

\[
\lim_{x \to c} f(x) = \infty \quad \text{and} \quad \lim_{x \to c} g(x) = L
\]

Then:

1. \(\lim_{x \to c} [f(x) \pm g(x)] = \infty\)
2. \(\lim_{x \to c} [f(x)g(x)] \begin{cases} \infty & \text{if } g(x) \to 0 \\ -\infty & \text{if } L \neq 0 \end{cases}\)
3. \(\lim_{x \to c} \frac{g(x)}{f(x)} = 0\)

Ex. \(\lim_{x \to 1^-} \frac{x^2 - 3x}{x - 1} \quad \text{constant neg #}
\quad \text{something goes to 0}
\quad \rightarrow -\infty\)
\((4)^2 - 3(4) = 16 - 12 = 4\)

Ex. \(\lim_{x \to 0} \left[1 + \frac{1}{x^2}\right] = \infty\)

Ex. \(\lim_{x \to -3^+} \sqrt{x^2 - 9} = \frac{-x}{x - 3} = \lim_{x \to -3^+} \frac{-1}{x - 3} \quad \text{something goes to 0} = -\infty\)

Recall: \(\lim_{x \to 0} f(x) = \infty\) really means that the limit does not exist because \(f(x)\) increases or decreases without bound.

Ex. \(\lim_{x \to 0^-} \ln|\cos(x)| = \text{DNE}\)
\(\cos(\frac{\pi}{2}) = 0\)

Ex. \(\lim_{x \to 0^+} \left(\frac{x}{x^2}\right) = 0 - \infty = -\infty\)