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Dr Lin
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Thm 2.4.9 (Properties of Continuity)

If f and g are continuous at $x=c$ and $b \in \mathbb{R}$,

Then bf, fg, $\frac{f}{g}$ ($g \neq 0$), and $f \pm g$ are also continuous at $x=c$.

Some continuous functions:

1) constant --- $f(x) = c$

2) polynomial --- $p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$

3) rational --- $\frac{p(x)}{g(x)}$, $g(x) \neq 0$

4) radicals --- $f(x) = \sqrt[n]{x} = x^{1/n}$

5) trig --- $\sin(x)$, $\cos(x)$, ...

6) exp & log --- a^x, e^x, $\ln(x)$

7) composition --- $f(g(x))$

Ex. $h(x) = x + e^x$ $\longmapsto$ $f(x) = x$ is a poly of order 1, so $f(x)$ continuous on $\mathbb{R}$

$g(x) = e^x$ continuous on $\mathbb{R}$

Then $h(x) = f(x) + g(x)$ is continuous on $\mathbb{R}$

Ex $h(x) = \frac{x^2+1}{\cos(x)}$ $\quad$ $\cos(x) = 0$ for $x = (2k+1)\frac{\pi}{2}$ for $k \in \mathbb{Z}$

= all odd multiples of $\frac{\pi}{2}$

Ex Find x-values at which $g(x)$ is not continuous if $g(x) = \begin{cases} \sin(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0 \end{cases}$

$\lim_{x\to 0} g(x) = \lim_{x \to 0} \sin(\frac{1}{x}) = \text{DNE}$

$g(0) = 0$

Since $\lim_{x\to 0} g(x) \neq g(0)$, $g(x)$ has a discontinuity at $x=0$.

Ex $g(x) = \begin{cases} x\sin(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0 \end{cases}$

$\lim_{x\to 0} g(x) = \lim_{x\to 0} x\sin(\frac{1}{x}) = 0$ (by squeeze Thm)

if $x \to 0^+$ $\quad -1 \le \sin(\frac{1}{x}) \le 1$

$-x \le x\sin(\frac{1}{x}) \le x$

same, so $-|x| \le x\sin(\frac{1}{x}) \le |x|$

if $x \to 0^-$ $\quad -1 \le \sin(\frac{1}{x}) \le 1$

$-x \ge x\sin(\frac{1}{x}) \ge x$

$\lim_{x\to 0} -|x| = 0$

same

$\lim_{x\to 0} |x| = 0$

Since $\lim_{x\to 0} g(x) = g(0)$, g is continuous at $x=0$.

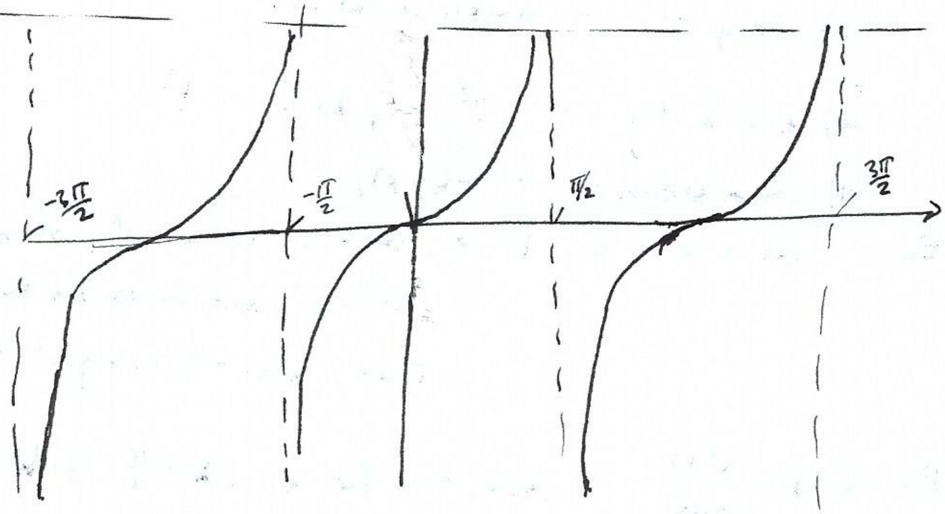
Since $x=0$ is the only problem point, g is continuous everywhere on $\mathbb{R}$.

Ex $f(x) = \tan(x) = \dfrac{\sin(x)}{\cos(x)}$

$\cos(x) \neq 0$

$\to x \neq (2k+1)\frac{\pi}{2}$

$\to$ non-removable discontinuity



Ex. $f(x) = \dfrac{x}{x^2-x} = \dfrac{x}{x(x-1)} = \dfrac{1}{x-1}$

$x=1$ non-removable discontinuity

$x=0$ removable discontinuity

$f(x)$ continuous everywhere on $\mathbb{R}$
except $x = 0, 1$

Then f continuous on $(-\infty, 0) \cup (0,1) \cup (1,\infty)$

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Ex. $g(x) = \dfrac{x}{x^2+1}$ $\qquad$ x continuous

x^2+1 continuous and never equal to 0 for $x \in \mathbb{R}$

Then $g(x)$ continuous on $\mathbb{R}$.

Ex. Find constants a, b so that $f(x)$ continuous on $\mathbb{R}$,

with $f(x) = \begin{cases} 2, & x \le -1 \\ ax+b, & -1 < x < 3 \\ -2, & x \ge 3 \end{cases}$

$*$ need to check continuity @ $x = -1, 3$

So we need $\lim_{x \to -1} f(x) = f(-1)$

and $\lim_{x \to 3} f(x) = f(3)$

$\lim_{x \to -1} f(x) = f(-1) = 2$

$\lim_{x \to -1^-} 2 = \lim_{x \to -1^+} (ax+b) = -a+b = 2$

$\to b = a + 2 = (-1) + 2 = +1$

$\lim_{x \to 3} f(x) = f(3) = -2$

$\lim_{x \to 3^+} (-2) = \lim_{x \to 3^-} (ax+b) = 3a+b = -2$

$= 3a + (a+2) = 4a + 2 = -2$

$4a = -4$

$a = -1$

So $f(x) = \begin{cases} 2, & x \le -1 \\ 1-x, & -1 < x < 3 \\ -2, & x \ge 3 \end{cases}$

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Intermediate Value Theorem

If f is continuous on $[a,b]$, $f(a) \ne f(b)$, and $f(a) < k < f(b)$, then there is a number c in $[a,b]$ such that $f(c) = k$.

Ex. Show that $f(x) = x^3 + 2x - 1$ has a zero on $[0,1]$.

$f(0) = 0^3 + 2 \cdot 0 + (-1) = -1$

$f(1) = 1^3 + 2 \cdot 1 - 1 = 2$

$\to f(a) \ne f(b)$

$f(x)$ is a polynomial $\to f$ continuous

$0 \in (-1, 2)$, so by IVT there exists $c \in [0,1]$ such that $f(c) = 0$