Math 2413
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24 Sep 2018
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Ex. Let $f(x)=x^2$ for $x\geq 0$ and let $f^{-1}(x)=\sqrt{x}$.
Show that the slopes of the graphs of f and f^{-1} are reciprocals at $(2,4)$ and $(4,2)$

$f'(x)=2x \qquad (f^{-1})'(x)=\dfrac{1}{2\sqrt{x}}$

$f'(2)=4 \qquad (f^{-1})'(4)=\dfrac{1}{4}$

$$(f^{-1})'(4)=\frac{1}{f'(f^{-1}(4))}$$

Thm: $(\arcsin x)'=\dfrac{1}{\sqrt{1-x^2}} \qquad (\arccos x)'=\dfrac{-1}{\sqrt{1-x^2}}$

$(\arctan x)'=\dfrac{1}{1+x^2} \qquad (\operatorname{arccot} x)'=\dfrac{-1}{1+x^2}$

$(\operatorname{arcsec} x)'=\dfrac{1}{|x|\sqrt{x^2-1}} \qquad (\arccos x)'=\dfrac{-1}{|x|\sqrt{x^2-1}}$

Ex Find the derivative with respect to x.

a) $\arcsin(\sqrt{x})+\operatorname{arcsec}(e^{2x}) \xmapsto{\frac{d}{dx}} (\arcsin(\sqrt{x}))'(\sqrt{x})'+(\operatorname{arcsec}(e^{2x}))'(e^{2x})'$

$$=\frac{\frac{1}{2\sqrt{x}}}{\sqrt{1-(\sqrt{x})^2}}+\frac{2\cancel{e^{2x}}}{|e^{2x}|\sqrt{(e^{2x})^2-1}}$$

*exponential function always positive, so $|e^{2x}|=e^{2x}$

$$=\frac{1}{2\sqrt{x}\sqrt{1-x}}+\frac{2}{\sqrt{e^{4x}-1}}$$

$$=\frac{1}{2\sqrt{x-x^2}}+\frac{2}{\sqrt{e^{4x}-1}}$$

b) $y=x\arccos(x)+\dfrac{\arcsin(3x)}{x}$

$$\frac{dy}{dx}=\left[\arccos(x)+\frac{-x}{\sqrt{1-x^2}}\right]+\left[\frac{-1}{x^2}\arcsin(3x)+\frac{1}{x}\cdot\frac{3}{\sqrt{1-(3x)^2}}\right]$$

$$=\arccos(x)-\frac{x}{\sqrt{1-x^2}}-\frac{\arcsin(3x)}{x^2}+\frac{3}{x\sqrt{1-9x^2}}$$

Ex Find the equation of the tangent line at the point $(1, \ln(\frac{\pi}{4}))$

for $x\arctan x = e^y$

$$\frac{d}{dx}\left[x\arctan x\right] = \arctan x + x\left(\frac{1}{1+x^2}\right)$$

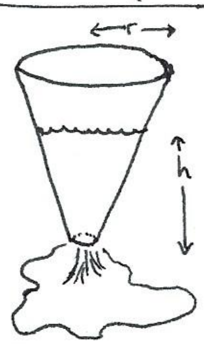
$$\frac{d}{dx}\left[e^y\right] = e^y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{\arctan(x) + \frac{x}{1+x^2}}{e^y} \xrightarrow{(1, \ln\frac{\pi}{4})} \frac{\arctan(1) + \frac{1}{1+1}}{e^{\ln\frac{\pi}{4}}} = \frac{\frac{\pi}{4} + \frac{1}{2}}{\frac{\pi}{4}}$$

$$= \frac{\pi+2}{\pi} = 1 + \frac{2}{\pi}$$

Then $y - \ln\frac{\pi}{4} = (1+\frac{2}{\pi})(x-1)$ is tangent to $x\arctan x = e^y$ at the point $(1, \ln\frac{\pi}{4})$

Related Rates



For example, when water drained from conical tank, the volume, radius, and height are all functions of time and are related by

$$V = \frac{\pi}{3}r^2 h$$

We use implicit differentiation to obtain the related-rate equation.

$$\frac{d}{dt}V = \frac{d}{dt}\left[\frac{\pi}{3}r^2 h\right]$$

$$= \frac{\pi}{3}\left[2r\left(\frac{dr}{dt}\right)h + r^2\left(\frac{dh}{dt}\right)\right]$$

$$= \frac{\pi}{3}\left[2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right]$$

Guidelines for solving related-rates problems:

1) Identify known and unknown quantities
2) write an equation involving the rates of change
3) use chain rule to differentiate implicitly with respect to time t.
4) Substitute known values for their variables and rates of change.
5) solve for desired rate of change

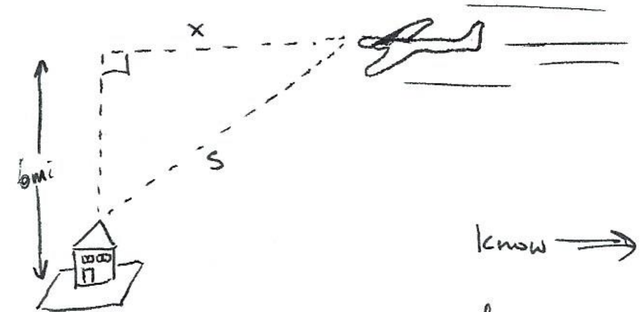
Ex. known: $\frac{dV}{dt} = 4.5 \frac{ft^3}{min}$ find: $\frac{dr}{dt}$ when $r = 2$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt}$

$= 4\pi r^2 \frac{dr}{dt} = 4.5 \longmapsto \frac{dr}{dt} = \frac{4.5}{4\pi r^2}$

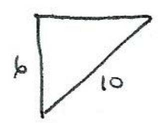
$\left.\frac{dr}{dt}\right|_2 = \frac{4.5}{16\pi}$



The distance s is decreasing at a rate of 400 $\frac{\text{miles}}{\text{hour}}$ when s = 10 miles. What is the speed of the plane?

know $\Longrightarrow$ when $s = 10$, $\frac{ds}{dt} = -400$

find $\Longrightarrow$ when $s = 10$, $\frac{dx}{dt} = ?$



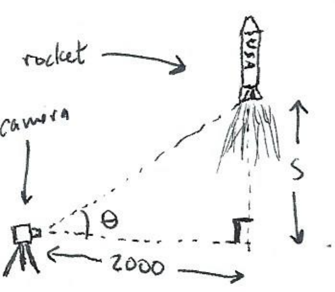
$x^2 + 6^2 = x^2 + 36 = s^2$

$2x\frac{dx}{dt} = 2s\frac{ds}{dt} = 2s(-400) = -800s$

$x\frac{dx}{dt} = -400s$

$\frac{dx}{dt} = \frac{-400(10)}{\sqrt{10^2 - 6^2}} = \frac{-4000}{\sqrt{100-64}} = \frac{-4000}{\sqrt{36}} = \frac{-4000}{6} = \frac{-2000}{3}$

$\Longrightarrow$ speed $= |v| = \left|\frac{dx}{dt}\right| = \left|\frac{-2000}{3}\right| = \frac{2000}{3}$



$S(t) = 50t^2$

Find $\left.\frac{d\theta}{dt}\right|_{t=10}$

$\tan\theta = \frac{s}{2000}$

$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{2000}\frac{ds}{dt}$

$\left(\frac{(2000^2 + s^2)^{1/2}}{2000}\right)^2 \frac{d\theta}{dt} = \frac{1}{2000}(100t) = \frac{t}{20}$

$\frac{d\theta}{dt} = \frac{t}{20}\left(\frac{(2000^2 + (50t^2)^2)}{2000^2}\right)^{-2} = \frac{t}{20}\left(1 + \left(\frac{50t^2}{2000}\right)^2\right)^{-2}$

$\left.\frac{d\theta}{dt}\right|_{t=10} = \frac{1}{2}\left(1 + \left(\frac{5000}{2000}\right)^2\right)^{-1} = \frac{1}{2}\left(1 + \frac{25}{4}\right)^{-1} = \frac{1}{2}\left(\frac{29}{4}\right)^{-1}$

$= \frac{2}{29}$