Ex. (cont)

$$h(x) = \ln\left[\frac{x(x^2+1)^2}{(2x^3-1)^{1/2}}\right] = \ln\left[x(x^2+1)^2\right] - \ln\left[(2x^3-1)^{1/2}\right]$$

$$= \ln(x) + 2\ln(x^2+1) - \frac{1}{2}\ln(2x^3-1)$$

$$h'(x) = \frac{1}{x} + 2\cdot\frac{1}{x^2+1}\cdot 2x - \frac{1}{2}\cdot\frac{1}{2x^3-1}\cdot 6x = \frac{1}{x} + \frac{4x}{x^2+1} - \frac{3x}{2x^3-1}$$

or

$$h'(x) = \left(\frac{x(x^2+1)^2}{(2x^3-1)^{1/2}}\right)^{-1}\cdot\left[\frac{\left(x(x^2+1)^2\right)'(2x^3-1)^{1/2} - x(x^2+1)^2\left((2x^3-1)^{1/2}\right)'}{2x^3-1}\right]$$

$$= \frac{(2x^3-1)^{1/2}}{x(x^2+1)^2}\cdot\frac{1}{2x^3-1}\left[\left((x^2+1)^2 + 2x(x^2+1)(2x)\right)(2x^3-1)^{1/2} - x(x^2+1)^2\left(\frac{1}{2}(2x^3-1)^{-1/2}(6x)\right)\right]$$

$$= \ldots$$

Ex. Find the equation of the line tangent to $f(x) = x^2(1-x^2)^{1/2}$ at $x = \frac{1}{2}$.

$$f'(x) = 2x(1-x^2)^{1/2} + x^2\frac{1}{2}(1-x^2)^{-1/2}(-2x) = 2x(1-x^2)^{1/2} - x^3(1-x^2)^{-1/2}$$

$$f'\left(\tfrac{1}{2}\right) = 2\left(\tfrac{1}{2}\right)\left(1-\left(\tfrac{1}{2}\right)^2\right)^{1/2} - \left(\tfrac{1}{2}\right)^3\left(1-\left(\tfrac{1}{2}\right)^2\right)^{-1/2} = 1\cdot\left(\tfrac{3}{4}\right)^{1/2} - \tfrac{1}{8}\left(\tfrac{3}{4}\right)^{-1/2} = \frac{\sqrt{3}}{2} - \frac{1}{8}\frac{2}{\sqrt{3}} = \frac{\sqrt{3}}{2} - \frac{1}{4\sqrt{3}}$$

$$f\left(\tfrac{1}{2}\right) = \left(\tfrac{1}{2}\right)^2\left(1-\left(\tfrac{1}{2}\right)^2\right)^{1/2} = \frac{1}{4}\left(\frac{3}{4}\right)^{1/2} = \frac{\sqrt{3}}{8}$$

Then the line $y - \frac{\sqrt{3}}{8} = \left(\frac{\sqrt{3}}{2} - \frac{1}{4\sqrt{3}}\right)\left(x - \frac{1}{2}\right)$ is tangent to $f(x) = x^2(1-x^2)^{1/2}$ at $x = \frac{1}{2}$

Def. If α is a positive real number, $\alpha \neq 1$, then the exponential function with base α, α^x, is defined by $\alpha^x = e^{\ln(\alpha)x}$,

and the logarithmic function with base α, $\log_\alpha(x)$, is defined by

$$\log_\alpha(x) = \frac{1}{\ln(\alpha)}\cdot\ln(x)$$

Thm. Let $\alpha \neq 1$ be a positive real number and let f be a differentiable function of x.

Then:

$$(\alpha^x)' = (\ln(\alpha))\cdot\alpha^x \qquad \left(\alpha^{f(x)}\right)' = (\ln(\alpha))\alpha^{f(x)}f'(x)$$

$$(\log_\alpha(x))' = \frac{1}{\ln(\alpha)}\cdot\frac{1}{x} \qquad \left(\log_\alpha[f(x)]\right)' = \frac{1}{\ln(\alpha)}\cdot\frac{1}{f(x)}\cdot f'(x)$$

$$m(x) = \log_{10}\left[\cos\left(\tfrac{x}{2}\right)\right] + 2^{3x} = \frac{1}{\ln(10)} \cdot \ln\left[\cos\left(\tfrac{x}{2}\right)\right] + 2^{3x}$$

$$m'(x) = \frac{1}{\ln(10)} \cdot \frac{1}{\cos(\frac{x}{2})} \cdot \left(-\sin\left(\tfrac{x}{2}\right)\right) \cdot \frac{1}{2} + \ln(2) \cdot 2^{3x} \cdot 3$$

$$= \frac{-\sin(\frac{x}{2})}{2\ln(10)\cos(\frac{x}{2})} + 3 \cdot 2^{3x} \cdot \ln(2)$$

Ex. Given that $\begin{cases} g(5) = -3 \\ g'(5) = 6, \end{cases}$ $\begin{cases} f(5) = 3 \\ f'(5) = -2 \end{cases}$ find $h'(5)$

a) $h(x) = f(x)g(x)$

$$h'(x)\Big|_5 = \Big[f'(x)g(x) + f(x)g'(x)\Big]_5 = (-2)(-3) + (3)(6) = 6 + 18 = 24$$

b) $h(x) = g^3(x)$ $\qquad h'(5) = 3(-3)^2(6) = 18 \cdot 9 = 162$

$h'(x) = 3g^2(x)g'(x)$

c) $$\ln\left[\left(\frac{3g(x)}{h(x)}\right)^2\right] = 2\ln\left(\frac{3g(x)}{h(x)}\right] = 2\Big[\ln(3) + \ln(g(x)) + (-1)\ln(h(x))\Big]$$

$h''(x)$

$$h'(x) = 2\left[0 + \frac{g'(x)}{g(x)} - \frac{h'(x)}{h(x)}\right] \qquad h'(5) = 2\left[\frac{6}{-3} - \frac{-2}{3}\right] = -\frac{8}{3}$$

Find $\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right)$

$$= \lim_{x\to\infty} \frac{\sin(\frac{1}{x})}{\frac{1}{x}} \quad , \text{ let } t = \frac{1}{x}$$

$$= \lim_{t\to 0} \frac{\sin(t)}{t}$$

$$= 1$$

* $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$

* $\lim_{x\to\infty} \frac{1}{x} = 0$