Ex. Find the equation of the tangent line of \( f(x) = \sqrt{x} \) at the point \((4,2)\). Discuss the behavior of \( f \) at \((0,0)\).

\[
f'(x) = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left[ f(x+\Delta x) - f(x) \right] = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left[ \sqrt{x+\Delta x} - \sqrt{x} \right]
\]

\[
= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left[ (\sqrt{x+\Delta x})^2 - (\sqrt{x})^2 \right] \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}} = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left[ \frac{x+\Delta x - x}{\sqrt{x+\Delta x} + \sqrt{x}} \right]
\]

\[
= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \quad \rightarrow \quad f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}
\]

\( y-2 = \frac{1}{4}(x-4) = \frac{1}{4}x - 1 \)

\( y = \frac{1}{4}x + 1 \) is the equation of the line tangent to \( f(x) = \sqrt{x} \) at \((4,2)\).

at \( x=0 \), \( f'(x) = \lim_{x \to 0} \frac{1}{2\sqrt{x}} = 0 \) \( \rightarrow \) \( f \) is not differentiable at \( x=0 \).

Ex. Find an equation of the line tangent to \( f(x) = \sqrt{x} \) and parallel to the line \( 2x-y+1=0 \).

\( 2x-y+1=0 \)

\( 2x+1 \rightarrow \) slope \( =2 \) = slope of given tangent line \( f'(x) = \frac{1}{2\sqrt{x}} \) for some \( x > 0 \).

\[
y - \frac{1}{4}x = 2(x - \frac{1}{16}) = 2x - \frac{1}{8}
\]

\( y = 2x - \frac{1}{8} + \frac{1}{8} = 2x + \frac{1}{8} \)

\( y = 2x + \frac{1}{8} \) is tangent to \( f \) at \((\frac{1}{8}, \frac{1}{16})\)

and is parallel to \( g(x) \).

Ex. Find the equation of the line tangent to \( f(x) = \sqrt{x} \) and perpendicular to the line \( 2x+y+1=0 \).

\( 2x+y+1=0 \)

\( 2x+1 \rightarrow \) slope \( = -2 \) = slope of given line \( \frac{1}{2} = \frac{1}{2\sqrt{x}} \) \( \rightarrow \) \( f'(x) = \frac{1}{2\sqrt{x}} \)

\[\frac{1}{2} = \frac{1}{2\sqrt{x}} \quad \rightarrow \quad \sqrt{x} = 1 = f(1) \]

\( x = 1 \)

\( y-1 = \frac{1}{2}(x-1) \)

\( y = \frac{1}{2}x + \frac{1}{2} = \frac{1}{2}(x+1) \)

\( y = \frac{1}{2}(x+1) \) is tangent to \( f(x) \) at \((1,1)\) and perpendicular to \( g(x) \).

* If two lines are perpendicular, the product of their slopes is \(-1\). *
If \( f \) is differentiable at \( x=c \), then \( f \) is continuous at \( x=c \).

If \( f \) is not continuous at \( x=c \), then \( f \) is not differentiable at \( x=c \).

**Determine whether the function is differentiable at the given point.**

1) \( f(x) = |x-2| \) at \( x=2 \)

\[
f'(c) = \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x}
= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left(|2+\Delta x-2| - |2-2|\right)
= \lim_{\Delta x \to 0} \frac{|\Delta x|}{\Delta x}
= \begin{cases} 1, & \Delta x > 0 \\ -1, & \Delta x < 0 \end{cases}
\]

Since \( f \) is not differentiable at \( x=2 \), there is a corner, which indicates that \( \lim_{x \to 2^+} f \neq \lim_{x \to 2^-} f \), which means \( \lim f \) does not exist.

Show numerically:
\[
\lim_{x \to 2^-} \frac{f(x)-f(2)}{x-2} = \frac{-0-0}{x-2} = -1
\]
\[
\lim_{x \to 2^+} \frac{f(x)-f(2)}{x-2} = \frac{(x-2)-0}{x-2} = 1
\]

Since \( \lim_{x \to 2^-} f = -1 \neq \lim_{x \to 2^+} f \), if \( f \) is not continuous at \( x=2 \), then \( f \) is not differentiable at \( x=2 \).

2) \( g(x) = x^{1/3} \) at \( x=0 \)

\[
\lim_{x \to 0} \frac{g(x)-g(0)}{x} = \lim_{x \to 0} \frac{1}{3} x^{-2/3} \quad \lim_{x \to 0} \frac{1}{x} \quad \lim_{x \to 0} \frac{1}{(x^{1/3})} = \infty
\]

Since the limit does not exist, the function is not differentiable at the point.

5) \( h(x) = \|x\| \)

Since \( h \) is not continuous at \( x=0 \), \( h \) is not differentiable at \( x=0 \).

4) \( l(x) = \frac{2}{x-3} \)

\( l \) is not continuous at \( x=3 \), so \( l \) is not differentiable at \( x=3 \).