Riemann Sums and Definite Integrals

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Let f be defined on $[a,b]$ and let Δ be a partition of $[a,b]$ given by

$a = x_0 < x_1 < x_2 < \dots < x_n = b$, where Δx_i is the width of the ith subinterval $[x_{i-1}, x_i]$.

If $c_i \in [x_{i-1}, x_i]$, then $\sum_{i=1}^{n} f(c_i)\Delta x_i$ is called a Riemann Sum of for the partition Δ.

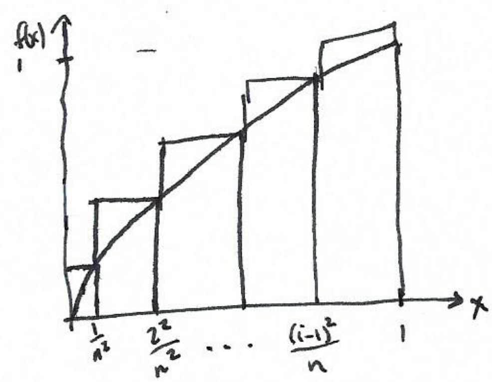
The norm $\|\Delta\|$ of the partition Δ is the width of the largest subinterval of a partition.

If every subinterval has equal length, the the partition is regular, and the norm is denoted by $\|\Delta\| = \Delta x = \frac{b-a}{n}$

For a general partition, $\frac{b-a}{\|\Delta\|} \le n$. In a regular partition, $(\|\Delta\| \to 0) \Leftrightarrow (n \to \infty)$

Ex $f(x) = \sqrt{x}$, $x \in [0,1]$. Evaluate $\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i$, where $c_i = \frac{i^2}{n^2}$ are the right endpoints and

$$\Delta x_i = \frac{i^2}{n^2} - \frac{(i-1)^2}{n^2} = \frac{2i-1}{n^2}$$



$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{i^2}{n^2}\right)\left(\frac{2i-1}{n^2}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i}{n} \cdot \frac{2i-1}{n^2} = \lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} (2i^2 - i)$$

$$= \lim_{n\to\infty} \frac{1}{n^3}\left[2 \cdot \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}\right] = \lim_{n\to\infty} \frac{(n+1)(2n+1)}{3n^2} - \frac{n+1}{2n^2}$$

$$= \frac{2}{3}$$

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Def. If f defined on $[a,b]$ and $\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = \lim_{n\to\infty} \sum f(c_i)\Delta x_i$ exists, then f is integrable on $[a,b]$ and the limit is denoted $\lim_{\|\Delta\|\to 0} \sum f(c_i)\Delta x_i = \int_a^b f(x)\,dx$.

This limit is called the definite integral of f from a to b. a and b are the lower and upper limits of integration.

Thm. If f is continuous on $[a,b]$ then f is integrable, so $\int_a^b f(x)\,dx$ exists.

Ex. Evaluate $\int_{-2}^{1} 2x\,dx$ by definition.

Choose $\Delta x_i = \Delta x = \frac{1-(-2)}{n} = \frac{3}{n} = \|\Delta\|$

right endpoints: $c_i = -2 + i\Delta x$
$= -2 + \frac{3i}{n}$

$$\int_{-2}^{1} 2x\,dx = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i =$$

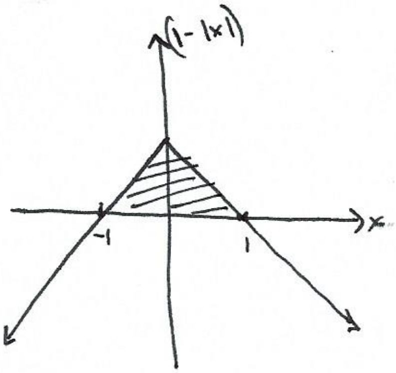
$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(-2+\frac{3i}{n}\right)\frac{3}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} 2\left(-2+\frac{3i}{n}\right)\frac{3}{n} = \lim_{n\to\infty} \frac{3}{n}\sum_{i=1}^{n}\left(-4+\frac{6i}{n}\right)$$

$$= \lim_{n\to\infty} \frac{3}{n}\left[-4\cdot n + \frac{6}{n}\cdot\frac{n(n+1)}{2}\right] = \lim_{n\to\infty}\left[-12 + 9\cdot\frac{n+1}{n}\right] = -12+9 = -3$$

Thm: If f is continuous and non-negative on $[a,b]$, then the area of the region bounded by f, the x-axis, and the vertical lines $x=a$ and $x=b$ is

$$\text{Area} = \int_a^b f(x)\,dx$$

Ex. $\int_{-1}^{1} (1-|x|)\,dx \longrightarrow$ Sketch the region corresponding to the definite integral. Evaluate the integral using geometric formula.



area of triangle $= \frac{1}{2}bh$

$$\int_{-1}^{1} (1-|x|)\,dx = \frac{1}{2}\cdot 2\cdot 1 = 1$$